INSTITUT POLYTECHNIQUE DE PARIS

ALGEBRAIC AND TOPOLOGICAL MODELS OF DIRECTED SYSTEMS

EXPLORING DIMENSIONS OF CALCULATION VIA ALGEBRAIC, CATEGORICAL, AND HOMOTOPICAL APPROACHES

Director: Eric Goubault
Co-director: Phillippe Malbos

Jury members

Lisbeth Fajstrup
Samuel Mimram
Damien Pous
Martin Raussen (referee)
Luigi Santocanale
Carlos Simpson (referee)

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Cameron Calk

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CHAPTER 0.
INTRODUCTION

This thesis consists in several mathematical approaches to calculation and computation. These concepts, central to mathematics and computer science, are often confounded. However, there is a natural distinction to be made between the two. Indeed, calculation is a constructive process which transforms one (or more) inputs into one (or more) outputs, while computation is the act of calculating in some well-defined (formal) context. In other words, computation is a concrete instantiation of the abstract principles of calculation.

Computation is, and has long been, a central tool in most scientific endeavours. A computation is a determination, a derivation, a formal method of transforming a fact or observation into a concrete result. It encapsulates the scientific desire to establish unique answers, thereby distinguishing or relating different phenomena. In this sense, it is a constructive approach to some notion of equivalence; it provides a formal method by which to cut things apart or group things together.

The mechanisms by which computations take place are governed by the maxims of calculation. Indeed, determining whether computation in some context provides, for any specified input, a unique answer, can be determined by studying the abstract system of calculation underlying the computations. The study of calculation as an abstract directed process therefore underlies computation, a fundamental tool of science. Due to this wide range of application, calculation has been studied in various branches of mathematics for centuries. Moreover, calculation has a wide range of interpretation: any directed system could be interpreted as some form of calculatory process.

Alongside providing a means of distinguishing or rendering certain observations or concepts equivalent, another boon of calculation is its constructive nature. We usually think of calculation as an algorithmic process, a series of steps which lead from a specification to an output. Approaching abstract calculation from a mathematical point of view, using tools from higher category theory or algebraic topology, lies on the exciting interface between computer science, constructive mathematics, and other classical mathematical frameworks.

In general, there are many ways one can reduce an object to another in a system of calculation. This ambiguity in calculation and thereby computation can be approached in many ways. Here, we distinguish two: on the one hand, we can view these as *choices* in the way we calculate, and on the other, we may choose to view them as *simultaneous* calculations. This ideological distinction brings us to the domains of abstract rewriting, string rewriting and normalisation theory in the first case, and concurrency theory, interleaving systems and consensus problems in the second.

Rewriting theory approaches the ambiguity of multiple possible reductions as a problem of choice. This is resolved first by considering what we will refer to as *consistency* properties. These express that the choices made while calculating do not affect the outcome, *i.e.* that the system of calculation is internally consistent, or that equivalent objects will produce the same unique answer when calculations are performed. Shortly put, this means that the system of calculation is consistent with an ambient notion of equivalence. These consistency properties express directed connectedness and zero-dimensional contractibility properties, as will be made explicit, and are expressed in the context of abstract rewriting systems (ARS). *Coherence* properties push this study of choice to higher dimensions, providing constructive methods for producing truly free systems of calculation.

In concurrency theory, calculation is considered in a different manner. Here, we consider systems of calculation which represent the simultaneous execution of several independent computations. Instead of a calculation being represented by a single path through a space of possible choices, concurrency theory tackles the problem of coordinating several distinct (deterministic) processes, each representing a concrete (linear) computation. *Synchronising* these independent calculations and understanding the properties of a such a concurrent system of calculation are central to this domain. These systems are thought of as spaces in which paths represent, not a choice, but a possibility. In this sense, the syntax of a concurrent system is captured spatially, and synchronisation problems can be related to the study of these *directed spaces*. As in classical topology, algebraic invariants are used to characterise, as well as understand properties of, such spaces.

Formalising coherence

Consistency is an important property for systems of calculation. It expresses determinacy of calculation, and a compatibility between the free closures of the system and a notion of equivalence. The study of coherence properties in the context of rewriting theory pushes this analysis up in dimension, considering equivalences between paths and defining (higher) systems of calculation underlying these equivalences.

Coherence is a notion from higher category theory describing algebraic axioms up to some notion of homotopy, which is encoded by higher dimensional cells. It may also be seen as a form of inductive definition: structure at a given level is encoded via structure at the level above, allowing a free description of the entire structure. This notion is widely used, both in abstract settings, but also in calculation.

The categorical notion of coherence was introduced independently by MacLane [88] and by Epstein [39]. In the case of the former, the objects of study are monoidal categories, in which the multiplicative structure is encoded by natural transformations describing unit and associativity laws. Any two expressions built from the monoidal structure which are equivalent modulo the monoid axioms must be related by a sequence of these natural transformations. However, these natural transformations are also a part of the structure. When two different sequences of natural transformations relate the same pair of terms, they must be equal in order for the whole structure to be coherent. In other words, any diagram of natural transformations must commute.

In homological algebra, higher coherence for modules ensures the existence of a free resolution of the considered algebraic structure [65]. This is also closely related to the homotopical notion of cofibrance in Quillen model structure interpretations of categorical structures [6, 96]. Indeed, a coherent structure has trivial homotopy in all higher dimensions. Determining the coherence of a given structure is therefore an important problem in categorical algebra.

Structural coherence is assured by a so-called *coherence theorem*, namely a result relating local coherence to global coherence. In the case of monoidal categories, MacLane showed that it is sufficient to assume that a small number of elementary diagrams commute to assure coherence of the entire structure [88]. Such theorems, relating local properties to global properties, are frequently used to prove coherence.

In the modern context of categorical algebra, structures are becoming increasingly dependent on complicated coherence checks, see for example [99]. Indeed, the natural notion of equivalence in categories is that of isomorphism rather than equality. This trend is followed in the domain of homotopy type theory[112], in which the contractibility of identity types, via higher types, is important. This idea has a natural link to calculation, since encoded therein is the fact that an equivalence is accompanied by a proof, a derivation, an isomorphism, providing a specific witness.

This calculatory angle provides a link to rewriting theory. Another is provided by the idea that coherence of certain elementary shapes determines coherence of all shapes. Indeed, a general schema in rewriting results is that a local property implies a global property. This is achieved by studying the properties of the directed system of calculation described by some abstract rewriting system. However, abstract rewriting theory is, in essence, one-dimensional. Therefore the study of coherence in this context stops at the level of consistency properties.

Abstract rewriting systems have been studied for over a century, first in a relation algebraic setting [8] and finally ending up in the general setting of categorical algebra, see for example [67]. The first appearance of determining coherence via rewriting, Squier's theorem for string rewriting [103, 104], uses one-dimensional properties to construct the coherent presentation. This results in a free, cofibrant replacement of the presented structure. Squier's constructions were formulated in the categorical language of polygraphs in [68] for monoids and in [65] for higher categories.

However, a new paradigm for abstract rewriting proofs has recently been developed. Indeed, Kleene algebra have been used to describe the properties of abstract rewriting systems, and classical theorems from the domain have also been stated and proved internally to Kleene algebras [28, 109], see also [110]. These algebras, originally developed in the domain of formal languages [23], are in particular a generalisation of relation algebras, which is where the link to rewriting becomes clear. Diagrammatic reasoning is summed up in a point-free, algebraic way, replacing deduction by calculation. Indeed, deductions follow a simple algebraic flow given by the structure of Kleene algebra. Equipping these algebras with domain and codomain operations leads to notions of termination, and confluence properties are encoded via semi-commutations in the algebra. Moreover, these algebras lend themselves naturally to formalisation in proof assistants and checkers, see for example the implementations [2] in the proof assistants Isabelle/HOL and [94] in Coq, both of which have applications in program verification.

However, formal tools for tackling coherence problems are lacking. Indeed, while the relational and Kleene algebraic framework for consistency proofs by rewriting have been established, similar frameworks for higher rewriting methods have not yet been presented. As categorical structures and type theories become more dependent on complicated coherence checks, such a framework would provide a means by which such verifications are automated.

In this thesis, we combine the higher dimensional approach to rewriting with this algebraic description of the mechanisms of calculation. Having introduced the structure of higher Kleene algebra in [17], we capture abstract rewriting theorems in all dimensions. Moreover, we formulated and proved the coherence theorem for abstract rewriting systems internally to these algebraic structures [16]. These higher Kleene algebra naturally appear as power-set liftings of higher categorical structures. This provides, on the one hand, a point-free algebraic description of the mechanisms of higher dimensional rewriting, and on the other, a means of formalising the proofs of coherence and consistency theorems in rewriting.

Invariants in directed topology

Algebraic invariants are used extensively in topology to provide a means of classifying the wide array of spaces which point-set topology births. These provide obstructions for equivalence; if two spaces have non-isomorphic invariants, they cannot be homeomorphic spaces. Indeed, this is how the term “invariant” is derived, in the sense that they correspond to properties of the space which are invariant under homeomorphism. A first example of an invariant for topological spaces is that of connected components, but classical algebraic invariants also have a spatial interpretation, for example the fact that homology generators correspond to “holes” in the space.

In the domain of concurrency, a semantics for distributed systems of calculation is given by the notion of *directed space*. For complete accounts of directed spaces and their relation to concurrency, we refer the reader to [43, 61]. These are topological spaces augmented by specifying a set of “permitted” paths in the space. These paths provide a notion of direction in the space: we may only move through the space along these specified trajectories. In concurrency theory, this directedness models the irreversibility of time. Indeed, since each path represents a possible execution of simultaneous computations, moving along such a path represents an action which cannot be undone.

The study of these spaces is quite different to the study of usual topological spaces. Firstly, (path) connectedness of points acquires a directed flavour. For example, moving from one place to another in a city in which all streets are unidirectional is a much harder problem than when one can travel down a street in either direction. Moreover, whereas the objects of study in point-set topology are typically the points, or sets of points, in the space, in the directed case we are interested in studying the properties of the directed paths in the context of the underlying space. This is largely due to the interpretation of paths as executions of a concurrent process: understanding the topology of the directed paths provides insight into the process it represents [95].

Directed algebraic invariants must therefore differ somewhat from their undirected counterparts, firstly as a consequence of the shift in objects of study. The complexity of the problem is augmented, since we must not only attempt to understand the relation between paths, but must do so while taking into account the topological features of the space they move through. It is no longer the connected components of the space, but those of path spaces, nor holes in the space, but obstructions between directed paths. These must be measured, while simultaneously providing a means of relating these measurements to extensions into the past or future of a path. A study of invariants for directed spaces may be found in [97].

Homology and homotopy, two classical invariants of topological spaces, have found directed counterparts in the form of *natural homology* and *natural homotopy*, respectively. These, developed in [31] and inspired by ideas from [97], encapsulate the relevant invariants for all spaces of parallel directed paths, but also the relations between them given by future and past extensions. The notion of homotopical equivalence of paths is consistent with the interpretation of directed spaces as semantics for concurrent processes: parallel paths which are homotopically related correspond to equivalent executions.

These invariants capture many features of directed spaces, but were found to be insensitive to *time-reversal* [87]. That is, natural homology and natural homotopy were not fine-grained enough to detect the reversal of all paths in the considered space. As invariants of directed spaces, this is not a desirable property. In this thesis, these invariants are refined with an algebraic ingredient relating concatenation of paths to products of the algebraic structures provided by homology or homotopy. We introduced this refinement of natural homotopy and homology and its application to time-reversal in [15].

Another desirable property of invariants is that they be computationally tractable. Indeed, one of the reasons mathematicians considered algebraic invariants for topological spaces is that a notion of computation is provided by the algebraic structure, thereby providing a constructive means to distinguish spaces. Classically, homology is the more tractable invariant of topological spaces. In the case of directed spaces, natural homology is, in practice, not computationally tractable. Indeed, not only must we calculate the homology of each directed path space, but also the homomorphisms between the resulting homology groups provided by past and future extensions.

The importance of these homomorphisms in the analysis of concurrent systems led us to consider relations between natural homology, as introduced in [32], and *persistent homology*, an application of homology in data analysis [19]. Persistence theory provides tractable methods for calculating uni-dimensional persistent homologies. In this thesis, we relate uni-dimensional persistence to directed spaces by considering extensions along directed paths and describe how this information may be amalgamated to recover the whole invariant. This provides an ideological link between these two applications of classical homology theory, while at the same time making a first step towards a more tractable invariant for concurrent systems as modelled by directed spaces.

Goal of the thesis

In both of the domains described above, abstract rewriting and concurrency theory, a (one dimensional) directed system expresses properties of some notion of calculation. Higher dimensional structure is employed to study the properties of calculation, or define invariants thereof. This thesis approaches the study of calculation in this optic.

Motivation. The objective of this thesis is to introduce a cadre of algebraic and topological formalisation of directed calculation.

However, before tackling this objective, we provide intuition about systems of calculation, both as rewriting and concurrent systems, for readers less familiar with these domains. This is treated in the Preamble, in which we describe calculatory processes in a general manner and provide a bird's eye view of the intuitions behind coherence on the one hand, and algebraico-topological invariants of concurrent systems on the other. It is included to provide a moral to the mathematical story recounted in Parts I and II. Since a non-trivial part of the pursuit of mathematics is in the development of ideas, I find this an important aspect to include.

Coherence and rewriting. In Part I, the problem of formalising the mechanisms of higher dimensional rewriting, and coherence, is tackled. This is accomplished by introducing a novel algebraic structure, higher Kleene algebra, first appearing in [17]. In particular, we provide a formal, algebraic formulation of the coherence theorem for abstract rewriting systems [16]. This provides a first step towards formalising coherence checks in categorical algebra.

Directed algebraico-topological invariants. The goal of Part II, on the other hand, is to refine algebraic invariants for directed spaces and provide first steps toward the tractable computation thereof. Equipping the pre-existing invariants natural homotopy and natural homotopy with an extra algebraic ingredient, we obtain a finer analysis of the directed space, capturing time-reversal via opposition. We then further enrich natural homotopy by introducing the notion of relative directed homotopy and proving the existence of a long exact sequence of natural homotopy functors. These refinements constitute the subject of [15]. Finally, the computation of natural homology is approached by relating this invariant to the domain of persistent homology. In this thesis we demonstrate how natural homology may be obtained as a colimit uni-dimensional persistent homology modules along traces in a partially ordered space.

STRUCTURE AND MAIN RESULTS

Preamble. The first part of this thesis is a description of the morals and intuitions behind the domains of rewriting and concurrency theory followed by short descriptions of the different models of calculation we study. Chapter 1 approaches the study of calculation, both in rewriting and concurrent paradigms, without the associated technicalities. This additionally serves to create the context for the main body of the thesis and in particular formulate the problems we address.

Chapters 2, 3, and 4 provide more grounded intuitions, instantiated in the models of calculation we study in this thesis. Chapter 2 describes the relational and algebraic aspects of calculation. In particular, the interpretation of algebraic properties as calculatory properties are discussed, first in the setting of abstract rewriting, then in modal Kleene algebra. The relation description given in the former is generalised in the latter. In Chapter 3, the same properties are described and discussed in the more spatial context of polygraphs, first uni-dimensionally, and then in a higher setting. The intuitions behind higher dimensional rewriting and coherence are also treated, providing context for the problems addressed in Part I. Finally, in Chapter 4 we discuss combinatorial and topological models of concurrent systems. This provides intuition and context for the problems we address in Part II.

Part I. As stated above, the goal of Part I is to provide a formal algebraic context for higher dimensional rewriting and coherence proofs.

First, in Chapter 5, we recall notions from rewriting theory, starting in Section 5.1 with the classical, relational paradigm of rewriting. We contrast this point-free algebraic approach to rewriting with that provided by directed graphs, also called 1-polygraphs, in Section 5.2. It is also at this point that we recall notions from the theory of 1-categories. Section 5.3 recalls the notion of string rewriting system and critical branchings. This thesis does not address the string rewriting paradigm, but we include it for completeness purposes and for contrast with the case of abstract rewriting. This section also includes 2-categorical preliminaries. Finally, Section 5.4 recalls notions from higher dimensional rewriting, in particular relating these higher systems to abstract rewriting, as well as recalling definitions of and notations for higher categories. In each setting, whether relational, 1- or n-polygraphic, we recall classic rewriting theorems, namely the Church-Rosser theorem [22], see Theorems 5.1.10 and Corollary 5.4.15, as well as Newman's lemma [92], see Theorems 5.1.9, and Corollary 5.4.14. These result in what we call consistency theorems, see Theorems 5.1.11 and 5.4.16.

Chapter 6 recalls the Kleene algebraic paradigm of abstract rewriting [28, 109]. First, in Section 6.1, we provide definitions and properties of modal Kleene algebras for completeness purposes. In particular, Section 6.1.17 recalls a novel approach to conversion in modal Kleene algebra, first appearing in [16]. In Sections 6.1.21 and 6.1.22 we provide models of modal Kleene algebras derived as relational algebras or liftings of 1-polygraphs, the latter first appearing in [17]. Section 6.2 recalls how rewriting properties and theorems are captured in modal Kleene algebra, in particular the Church-Rosser theorem [109], see Theorem 6.2.10, Newman's lemma [28], see Theorem 6.2.9.

Next, Chapter 7 recalls coherence proofs via rewriting in the spirit of [68]. First, in Section 7.1 the coherence theorem for abstract rewriting systems represented by 1-polygraphs is recalled, and two proofs of the coherence theorem in this context are given, see Theorem 7.1.16. In particular, we recall the notion of strategy [67]. Section 7.2 briefly contrasts this with the string rewriting approach. Finally, in Section 7.3 the abstract coherence theorem is formulated in the higher dimensional setting, and a novel link is established between these higher systems and the coherence problem for abstract rewriting systems, see Theorem 7.3.6.

Chapter 8 is the first wholly original chapter of this thesis. The goal is to provide a succinct account of the coherence theorem in the two-dimensional via Kleene-algebraic structures. First, in Section 8.1, we recount a novel "vertical" approach to coherence, in order to relate the polygraphic paradigm of coherence to that of higher Kleene algebra. We also prove vertical versions of the Church-Rosser theorem and Newman's lemma, see Theorems 8.1.4 and 8.1.7. In Section 8.2 we introduce the notion of globular 2-Kleene algebra [17] and briefly describe their structural properties and the model given by 2-polygraphs, see Proposition 8.2.11, or Section 9.1.19 for more details. Next, Section 8.3 recounts the use of these algebras as a paradigm for coherence proofs with rewriting methods [17]. In Section 8.4, we formalise the notions of section and rewriting strategy, showing that a strategy corresponds to any skeleton of the iterative exhaustion. We then prove that confluences in the iterative exhaustion of an element correspond modally to the equivalence it generates, see Lemma 8.4.4. In Propostion 8.4.5 we prove that this novel definition of strategy has the desired properties with respect to normal forms and transitive closure. These results, as well as the definitions from this section, first appeared in [16].

Finally, in Section 8.5 we prove the abstract coherence theorem in the context of globular 2-Kleene algebras. First, we prove a normalising, strategic Newman's lemma:

8.5.1 Theorem (Coherent normalising Newman's lemma [16]). *Let K be a Boolean globular 2-Kleene algebra such that*

i) $(K_0, +, 0, \odot_0, 1_0, \neg_0)$ *is a complete Boolean algebra,*

ii) K_1 *is continuous with respect to* 0*-restriction, that is for all* $\psi, \psi' \in K_1$ *and* $(p_\alpha)_\alpha \subseteq K_0$ *we have* $\psi \odot_0 \vee p_\alpha \odot_0 \psi' = \vee\,(\psi \odot_0 p_\alpha \odot_0 \psi')$.

Let $\phi \in K_1$ *be convergent and* σ *be a skeleton of* $exh(\phi)$*. If* A *is a local confluence filler for* $(\overline{\phi}, \phi)$*, then* $|\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}) \geq \overline{\phi}^{*_0} \odot_0 \phi^{*_0}$.

As a direct consequence, we obtain the main theorem of this chapter, stated below. We consider an element A of a 2-Kleene algebra, which can be thought of as a set of two-dimensional directed tiles. By hypothesis, this element a local confluence filler for a (one-dimensional) element ϕ, as defined in Section 8.3.1, *i.e.* the tiles in A fill the local confluence diagrams associated to ϕ, which is thought of as a set of transitions or rewriting rules. A strategy σ, as defined in Section 8.4, associated to a convergent element ϕ is thought of as a skeletal subset of normalising paths. The conclusion of the theorem, an inequality in a 2-Kleene algebra, states that the equivalence generated by ϕ, denoted by $\phi^{\top}$, is included in the pre-image of confluences in the strategy under the completion $\hat{A}^{*_1}$ of A, the latter being defined in Section 8.3.2. In short, this means that every zig-zag in ϕ is paved to a confluence in σ by glueing the (directed) tiles in A:

8.5.2 Theorem (Abstract coherence theorem [16]). *Let K be a Boolean globular 2-Kleene algebra satisfying the additional hypotheses in Theorem 8.5.1 and $\phi \in K_1$ convergent. Given a normalisation strategy σ and a local confluence filler A for $(\overline{\phi}, \phi)$, we have*

$$|\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}) \geq \phi^{\top_0} = (\phi + \overline{\phi})^{*_0}.$$

Chapter 9 generalises this to higher dimensions, while providing a more in-depth treatment of the algebraic structure of higher Kleene algebra. Section 9.1 introduces n-Kleene algebras and their variants. In particular, a full account of the polygraphic model for these structures may be found in Section 9.1.19, and proving in Proposition 9.1.20 that the power-set lifting of a free n-category is indeed a higher Kleene algebra. These definitions appear in [17].

In Section 9.2 we first provide a full account of coherent rewriting in higher Kleene algebras, and then prove coherent versions of the Church-Rosser theorem, that is, with higher witnesses. First we give a proof by external induction, see Proposition 9.2.8 [17], and then using the notion of induction internal to Kleene algebras, *i.e.* via the Kleene-star:

9.2.9 Theorem (Coherent Church-Rosser in globular n-MKA [17]). *Let K be a globular n-modal Kleene algebra and $0 \leq i < j < n$. Given $\phi, \psi \in K_j$ and an i-confluence filler $A \in K$ of (ϕ, ψ), we have*

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq (\phi + \psi)^{*_i},$$

where $\hat{A}$ is the j-dimensional i-whiskering of A. Thus $\hat{A}^{*_j}$ is an i-Church-Rosser filler for (ϕ, ψ).

In Section 9.3 we describe the notion of termination in this higher algebraic setting and prove a coherent version of Newman's lemma:

9.3.2 Theorem (Coherent Newman's lemma for globular p-Boolean MKA [17]). *Let K be a globular p-Boolean modal Kleene algebra, and $0 \leq i \leq p < j < n$, such that*

i) *$(K_i, +, 0, \odot_i, 1_i, \neg_i)$ is a complete Boolean algebra,*

- **ii)** *K_j is continuous with respect to i-restriction,* i.e. *for all $\psi, \psi' \in K_j$ and every family $(p_\alpha)_{\alpha \in I}$ of elements of K_i such that $sup_I(p_\alpha)$ exists, we have*

$$\psi \odot_i sup_I(p_\alpha) \odot_i \psi' = sup_I(\psi \odot_i p_\alpha \odot_i \psi').$$

Let $\psi \in K_j$ be i-Noetherian and $\phi \in K_j$ i-well-founded. If A is a local i-confluence filler for (ϕ, ψ), then

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq \phi^{*_i}\psi^{*_i},$$

i.e. *$\hat{A}^{*_j}$ is a confluence filler for (ϕ, ψ).*

Section 9.4 presents the abstract coherence theorem for higher Kleene algebras as a consequence of that proved in Chapter 8, after generalising the notion of strategy to this setting:

9.4.3 Theorem (Abstract coherence theorem for HKA). *Let K be a p-Boolean globular n-Kleene algebra satisfying the additional hypotheses in Theorem 9.4.2 and $\phi \in K_j$ convergent. Given a normalisation strategy σ for ϕ and a local i-confluence filler A for $(\overline{\phi}, \phi)$, we have*

$$|\hat{A}^{*_j}\rangle_j(\sigma \odot_i \overline{\sigma}) \geq \phi^{\top_i} = (\phi + \overline{\phi})^{*_i}.$$

Finally, Section 9.5 checks the consistency of the above theorems with those in the polygraphic paradigm via the model described in Section 9.1.19. In particular, we prove the following:

9.5.5 Proposition ([17]). *With $\Gamma' := (\Gamma^c)^{*_n}$, the following equivalences hold:*

- **i)** *Γ is a (local) confluence filler for P $\iff$ Γ' is a (local) $(n-1)$-confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$,*
- **ii)** *Γ is a Church-Rosser filler for P $\iff$ Γ' is an $(n-1)$-Church-Rosser filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$.*

and use it to obtain Theorems 9.5.6 and 9.5.7, which state that the coherent, Kleene algebraic versions of Newman's lemma, Theorem 9.3.2 and the Church-Rosser theorem, Theorem 9.2.9, correspond to their polygraphic counterparts.

The final chapter describes a work in progress between myself, P. Malbos, D. Pous and G. Struth, based on [14, 41]. We describe the notions of catoid and modal quantale in Sections 10.1 and 10.2 before treating their n-dimensional analogues in Sections S:2-lr-msg and 10.4, respectively. (Higher) catoids generalise (higher) categories, while (higher globular) quantales constitute a special case of (higher globular) Kleene algebras. Finally, Section 10.5 presents a correspondence theorem for (higher) power-set quantales, providing a formal justification for the axiomatisation of higher Kleene algebras and their use in the domain of higher dimensional rewriting:

10.5.1 Theorem (Correspondence theorem for power-set n-quantales).

- **i)** *Let X be a local n-catoid. Then $(\mathcal{P}X, \subseteq, \odot_i, E_i, \ell_i, r_i)_{0 \leq i < n}$ is an n-quantale.*

ii) *Let $\mathcal{P}X$ be an n-quantale in which $E_i \neq \emptyset$. Then X is a local n-catoid.*

From this we deduce that every n-category lifts to an n-quantale, see Corollary 10.5.2.

Part II. The objective of Part II is twofold. Firstly, we address the problem of time-reversal invariance of natural homotopy and natural homology, solving it by adding natural structure to the invariants. Secondly, we link persistent homology to natural homology, establishing an ideological link between these two theories and showing that we can recover natural homology, an untractable invariant, from uni-dimensional persistent homologies.

Chapter 11 provides preliminaries for completeness purposes. First, in Section 11.1, we recall the notion of group object and the fixed object slice category, before describing the structural properties of group objects therein [93], see Section 11.1.3. We then recall the definitions of natural systems, first appearing in the cohomology theory of small categories [5], and an augmentation thereof called lax systems in Section 11.2. The latter are due to Porter [93], and combine the notion of natural system with the structure of a lax functor. Section 11.3 describes how this extra structure defines a composition pairing on a natural system, and that lax systems are equivalent to these objects. Tying all of this together, in Section 11.4 we show that natural systems with composition pairings are equivalent to group objects in the corresponding fixed object slice category.

In Section 11.5, we recall from [62] the definitions of semi-exact and homological categeories, exact sequences and show that the categories of groups and of pointed sets can be embedded in the category of actions, thus providing a common codomain for natural homotopy functors of all dimensions. This results in Proposition 11.5.7, a consequence of a result from [62], in which we show that we obtain a long exact sequence in the category of actions. In Section 11.6, we recall notions from directed topology. For this, we first recall the notion of directed space, see [43, 61], and then define the invariants, first introduced in [31]. In particular, we recall the trace category $\overrightarrow{\mathbf{P}}(\mathcal{X})$ associated to a directed space $\mathcal{X}$ and the n^{th} natural homotopy and natural homology functors, denoted by $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$, respectively. Finally, in Section 11.7, we recall the basics of persistence theory [19].

Chapter 12 contains the first original material in this part of the thesis. Its goal is to solve the problem of time-reversal invariance, which is outlined in an introductory section. In Section 12.1, we apply the results from Section 11.1 and 11.2 to the natural systems defined by the natural homotopy and homology functors. This results in Theorems 12.1.3 and 12.1.4, which state that we may augment natural homotopy with an extra algebraic ingredient: a composition pairing.

Putting together these theorems, as well as the correspondence recalled in Section 11.4 from [93], we obtain Theorem 12.1.5, which relates natural homotopy to group or split objects in $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$, the category of categories above $\overrightarrow{\mathbf{P}}(\mathcal{X})$ which preserve its elements. This provides an interpretation of natural homotopy, a functor, as a category above $\overrightarrow{\mathbf{P}}(\mathcal{X})$ with the same elements. We obtain a corresponding result for natural homology.

In Section 12.2 we begin by formally defining the time-reversal of a dispace $\mathcal{X}$, denoted by $\mathcal{X}^\sharp$, in Section 12.2.1, and then in Section 12.2.2 we define the notion of (strong) time-reversal of functors $\mathbf{dTop} \to \mathbf{Cat}$ with respect to opposition in $\mathbf{Cat}$. We then show that without composition pairings, the natural homotopy and homology functors associated to a dispace do not detect time-reversal. This time-symmetry of the original invariants is the subject of Section 12.2.3.

Section 12.2.4 contains the main theorems of this chapter, namely Theorems 12.2.5 and 12.2.6. These express that when equipped with a composition pairing, the invariants capture are strongly time-reversal, *i.e.* that we capture time-reversal by considering the opposite category of $\mathcal{C}^n_{\mathcal{X}}$ obtained via Theorem 12.1.5. In short, we show that as group or split objects in $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$, natural homotopy captures time-reversal via opposition:

12.2.5 Theorem ([15]). *Given a dispace $\mathcal{X} = (X, dX)$, $\mathcal{C}^n_{\mathcal{X}^\sharp}$ and $(\mathcal{C}^n_{\mathcal{X}})^o$ are isomorphic in $\mathbf{Gp}(\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$ for all $n \geq 2$, and in $\mathbf{Split}(\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$ for $n = 1$. In particular, the functors $\mathcal{C}^n_{-}$ are time-symmetric for all $n \geq 1$.*

12.2.6 Theorem ([15]). *For any $n \geq 0$, the functor $\mathcal{C}^n_{-} : \mathbf{dTop} \to \mathbf{Cat}$ is strongly time-reversal.*

We conclude Section 12.2 defining a notion of time-reversal relative to the category $\mathbf{opNat}(\mathbf{Act})$ of natural systems of actions and proving Theorem 12.2.8, which states that time-reversal of a functor $\mathbf{dTop} \to \mathbf{Cat}$ is equivalent to the notion for $\mathbf{opNat}(\mathbf{Act})$.

Finally in Section 12.3, we focus on further enriching natural homotopy by defining a notion of relative natural homotopy. In particular, we prove Theorem 12.3.2, which states that a long exact sequence of homotopy groups may be constructed from a pair $(\mathcal{X}, \mathcal{A})$ of dispaces.

12.3.2 Theorem ([15]). *Let $\mathcal{X}$ be a dispace and $\mathcal{A}$ be a directed subspace of $\mathcal{X}$. There is an exact sequence in $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathbf{Act})$:*

$$\cdots \to \overrightarrow{P}_n(\mathcal{A}) \to \overrightarrow{P}_n(\mathcal{X}) \to \overrightarrow{P}_n(\mathcal{X}, \mathcal{A}) \xrightarrow{\partial_n} \overrightarrow{P}_{n-1}(\mathcal{A}) \to \cdots$$
$$\cdots \to \overrightarrow{P}_2(\mathcal{A}) \xrightarrow{v} \overrightarrow{P}_2(\mathcal{X}) \xrightarrow{f} (\overrightarrow{P}_2(\mathcal{X}, \mathcal{A}), \overrightarrow{P}_2(\mathcal{X})) \xrightarrow{g} \overrightarrow{P}_1(\mathcal{A}) \xrightarrow{h} \overrightarrow{P}_1(\mathcal{X}) \to \overrightarrow{P}_1(\mathcal{X}, \mathcal{A}) \to 0.$$

We apply this to the special case of fibrations, resulting in Theorem 12.3.5 [15].

The final chapter of Part II, Chapter 13, addresses thee problem of calculating natural homology. It contains original contributions from ongoing work. This is a very large invariant, limiting its practicality. We study links between natural homology and persistent homology, the latter being tractable in most cases. First, in Section 13.1, we consider an example in order to illustrate the problem of obtaining filtrations from directed spaces without using traces. In Section 13.2, we show that natural homology is in fact a persistence object when considering partially ordered spaces, a class of directed space including cases of the most practical interest. This is essentially due to the observation that the factorisation category of the trace category of a partially ordered space is a poset:

13.2.4 Proposition . *For a pospace $\mathcal{X}$, $\mathbb{P}(\mathcal{X})$ is isomorphic to $F\overrightarrow{\mathbf{P}}(\mathcal{X})$.*

Next, we Proposition 13.2.6, which states that each trace yields a uni-dimensional persistent homology module. We apply this to the motivational example in Section 13.2.7 before turning to the question of amalgamating this uni-dimensional information. This is addressed in Section 13.3, but first we prove some results concerning colimits of posets, the first of which, Proposition 13.3.2, is folkloric, stating that a poset is the colimit of its chains, whereas the second Proposition 13.3.3, refines this for maximal chains. Next, in Section 13.3.4, we apply these colimit constructions to functors whose domains are posets and whose codomains are a fixed category $\mathcal{C}$. This gives the following result:

13.3.5 Proposition . *Let P be a poset, G a diagram whose colimit is P, $D : P \to \mathcal{C}$ a functor and F the diagram of functors given by restricting D along G. If $\mathcal{C}$ is poset co-complete, we have*

$$\operatorname*{colim}_{Pers(\mathcal{C})} F = D.$$

As direct consequences of the above propositions, we obtain the main theorem of this chapter, which states that natural homology for pospaces is recovered as a colimit of persistence modules in various ways:

13.3.7 Theorem . *Let $\mathcal{X} = (X, dX)$ be a pospace and α a point in X.*

- **i)** *The natural homology of $\mathcal{X}$ is the colimit in $Pers(\mathbf{Vect}_{\mathbb{K}})$ of the persistent homology along each of its traces.*
- **ii)** *The natural homology of $\mathcal{X}$ is the colimit in $Pers(\mathbf{Vect}_{\mathbb{K}})$ of the persistent homology its of its maximal traces, seen as chains in $\mathbb{P}(\mathcal{X})$, completed with pullbacks (resp. quasi-pullbacks).*
- **iii)** *The natural homology of the up-set of α, seen as a constant trace, in $\mathbb{P}(\mathcal{X})$ is the colimit in $Pers(\mathbf{Vect}_{\mathbb{K}})$ of the persistent homologies of the traces passing through α or of the maximal chains passing through α completed with pullbacks or quasi-pulbacks.*

PREAMBLE

Chapter 1.

Calculation, direction and choice

Calculation is ubiquitous in mathematics and computer science. The very first thing one learns in school about mathematics is how to compute sums. Formally, this consists in taking an expression such as $2+5+4$, *i.e.* a term built with natural numbers and an addition symbol $+$, and reduce it to a single number: 11. Hidden in this first exposure to mathematics lie the formal precepts of calculation.

First, we understand that when calculating with natural numbers, a notion of *direction* is involved. Indeed, while it is conventional in grade school to use the equality symbol $=$, calculation is directed: we should follow equalities toward a term of size one. This suggests that we are implicitly using something other than equality, namely a directed relation $\to$.

Next, we learn that the choices we make while calculating do not affect the final outcome. Indeed, we observe that first adding 2 and 5, and then adding 4 to their sum is equivalent to first adding 5 and 4, and then summing the result with 2. The situation may be summed up by the following diagram:

$$\begin{tikzcd}
 & 2+5+4 \arrow[dl] \arrow[dr] & \\
7+4 \arrow[dr] & & 2+9 \arrow[dl] \\
 & 11. &
\end{tikzcd}$$

Since calculations flow along the directed relation $\to$, we call the top part of the diagram a *branching* and the bottom part a *confluence*, terminology borrowed from the topology of rivers. The fact that these two calculations produce the same result can be seen as a consequence of associativity of addition, but also reflects an important calculatory property. Indeed, if every branching is confluent, the choices we make when calculating do not affect the final outcome, *i.e.* calculation is deterministic. As stated above, we are asked to always reduce the size of the term. This reflects the property of *termination*, namely that our calculations will eventually end and an answer may be given.

Finally, we understand that this means that any two terms which lead to the same answer are equal. This is somewhat obvious, since we know what equality means. However, in terms of calculation, we find that in order to show that $2+5+4$ is equal to $2+7+1+1$, it suffices to show that our calculations toward a term of size one both end up at the same answer. That is, it is not necessary to exhibit equalities in the style of

$$2+5+4=2+5+2+2=2+7+2=2+7+1+1,$$

but rather perform calculations of the form

$$2+5+4 \to 11 \leftarrow 2+7+1+1.$$

This means that our mode of calculation is *consistent* with the notion of equality.

Of course, these facts are not made explicit to children, already coping with their first experience of mathematical abstraction. However, these basic principles of calculation are central to the theory of *abstract rewriting*, a formal description of calculation.

The whole class is performing these calculations *at the same time.* In an ideal world, each of the students performs the steps and reaches the desired answers before the bell rings. In practice however, certain students may get stuck, ask for help, or even try to glean the answer from their neighbour's sheet. Two students sitting next to each other may both be stuck while the teacher is busy, and each is waiting for the other to perform the calculation in order to copy it on their own sheet. This leads to *deadlock*, a situation in which calculation stops.

Further, if a student produces an error it could be replicated by others copying the incorrect answer. This is of course remedied in exam conditions by making sure that students cannot see each other's sheets, making the calculations *mutually exclusive*, in that for the duration of the test, the students cannot access each other's written results.

This is our first exposure, whether we realise it or not, to *concurrency.* This is a domain of theoretical computer science in which calculations, or more generally actions, are being performed by independent agents *simultaneously.* Such a situation is again modelled in a directed way. Indeed, once an action has been performed, it cannot be undone. This means that, starting at some state of the system, we can only move to those states which lie in the *direction* given by elapsing time.

In this thesis, we study both types of directed systems, namely those arising from systems of calculation and those resulting in the study of simultaneous actions from independent agents.

1.1. ABSTRACT TRANSFORMATIONS

When considering calculation in an abstract setting, we no longer care what objects we are performing calculations on, a set X suffices, nor what the calculation steps actually do, relying instead upon some notion of transformation between elements of X.

1.1.1. Equivalence and calculation. Calculation usually relates to a notion of *equivalence*. Indeed, calculation is the answer to the indeterminacy of equivalence in the following sense. An equivalence relation on a set partitions the set into *equivalence classes*. These are sets of elements which should be thought as "the same element" under the considered notion of equivalence. However, when comparing elements of X, equivalence gives us no tools to choose how, if possible, to move from one to the other via equivalent elements. Using a directed relation whose symmetric, transitive and reflexive closure (see Section 5.1) corresponds to the considered equivalence relation provides a constructive approach to this quandary.

We distinguish the equivalence underlying the system of calculation and the syntactic equality in X notationally. When x and y denote the same element of X, *i.e.* are syntactically equal, we write $x = y$. When x and y are related by the notion of equivalence underlying the system of calculation, we write $x \equiv y$.

1.1.2. Direction. The transformations given by the system of calculation express equality, but their directedness allows us to move through the set X and produce an answer. If this were not the case, we would perhaps make choices that lead us *further* from an output than *closer*. For example, consider x and y in X such that we can transform x into y and vice versa. This results in an infinite loop of unhelpful calculations.

1.1.3. Calculation as transition. An *(abstract) system of calculation on* X is the given of a set of *rules* which tell us how to *reduce* elements of X to others. Due to the directedness of these rules, we often notationally denote the phrase "x is reduced to y" by the symbols

$$x \to y.$$

We say that x is the *source* of the *reduction step* and that y is its *target*. These rules *generate* sequences of calculations, which we call *reduction sequences*. Any calculation allowed by the system corresponds to such a sequence. For example, to reduce $2 + 5 + 4$ to 11, we first apply the rule $2 + 5 + 4 \to 2 + 9$ followed by the rule $9 + 2 \to 11$. If there exists a reduction sequence from x to y, we write

$$x \overset{*}{\to} y$$

to distinguish this from a single, atomic calculation. We again say that x (resp. y) is the *source* (resp. *target*) of the reduction sequence.

Note that every reduction step is a reduction sequence, but the converse is not true. Furthermore, we will also write $x \overset{*}{\to} y$ when $x = y$, *i.e.* x and y are the same element of X. The reduction sequences therefore correspond not only to (non-trivial) sequences of rules, but also include the syntactic equality in X.

1.1.4. Spatial interpretation of calculation. We now also have a spatial interpretation of our system: elements of X are thought of as points or nodes, while reductions are thought of as directed edeges between them. A reduction sequence then corresponds to a directed path in the space. When we forget about the directedness of steps and paths, we recover the equivalence relation underlying the system of calculation, its equivalence classes being given by connected components.

1.1.5. Concurrent computation. Turning to the notion of simultaneous calculation studied in the domain of concurrency, the abstract transformations we study have a different interpretation. When studying a system of independent agents performing actions at the same time, we may consider the *state* of the system at a given time. The state corresponds to which agents have performed which actions before the considered moment.

Instead of being reduction rules, the transformations in concurrency theory are interpreted as a transition from one state to another. Because of the irreversibility of an action, these transformations are again *directed*.

One of the main applications of concurrency theory is in the domain of *parallel programming*. This either involves time-sharing on a single processor, that is a single processor performing several tasks at the same time, or several processors performing actions independently, but sharing the same memory. In both cases, errors can occur if these actions are not coordinated. A simple example consists in two independent threads allocating different values to a common variable in parallel; the end result is not deterministic.

1.2. THE PROBLEM OF CHOICE

When calculating, we may be confronted with several choices. The properties of the considered system with respect to these choices is central to the abstract study of calculation. Given an abstract system of calculation $\to$ on a set X, we may have several choices of how to reduce a given element x via reduction sequences:

$$\begin{tikzcd}
 & & x \arrow[lld, bend right, "*"'] \arrow[ld, "*"'] \arrow[d, phantom, "\cdots"] \arrow[rd, "*"] \arrow[rrd, bend left, "*"] & & \\
x_1 & x_2 & \cdots & x_{k-1} & x_k.
\end{tikzcd}$$

This situation represents a conflict in the system: from a single element, several distinct answers are valid reductions. These *k-fold branchings* will be studied when considering coherence properties of the system, but a simpler form of conflict suffices for a first study of the consistency of the system. First, we will treat the simplest case, namely that of binary choice.

1.2.1. Branchings. A *branching* of the system is represented by the following diagram:

$$\begin{tikzcd}
 & x \arrow[ld, "*"'] \arrow[rd, "*"] & \\
x_1 & & x_2
\end{tikzcd}$$

We say that x is the *source* of this branching; it is the element at which the conflict takes place. This represents a non-deterministic behaviour, which is not desirable in a system of calculation.

While studying the branchings in a system is important, a recurring theme in the abstract study of calculation is taking *local* properties of the system to *global* properties. For this reason, we may be led to study *local branchings.* These are branchings of the form

$$\begin{tikzcd}
 & x \arrow[dl] \arrow[dr] & \\
x_1 & & x_2,
\end{tikzcd}$$

that is branchings in which both forks of the conflict are reduction steps, rather than sequences. We will see that consistency proofs allow us to go from resolution of local branchings to resolution of (global) branchings.

1.2.2. Branchings and equivalence. An abstract system of calculation is underlying an equivalence relation. What this effectively means in terms of calculations, is that whenever two (distinct) elements x and y are equivalent, *i.e.* $x \equiv y$, either $x = y$ or there exists a *zig-zag sequence* of calculations connecting them, as depicted below:

$$\begin{tikzcd}
 & x_1 \arrow[dl, "*"'] \arrow[dr, "*"] & & x_3 \arrow[dl, "*"'] \arrow[dr, "*"] & & \cdots & & x_{k-2} \arrow[dl, "*"'] \arrow[dr, "*"] & & x_k \arrow[dl, "*"'] \arrow[dr, "*"] & \\
x & & x_2 & & {} & \cdots & {} & & x_{k-1} & & y.
\end{tikzcd}$$

This means that the underlying notion of equivalence is given by a sequence of branchings in the system of calculation. To simplify notation, we will write $x \overset{*}{\leftrightarrow} y$ to denote that x and y are equivalent, that is, either $x \equiv y$ or are the same element of X, *i.e.* $x = y$.

1.2.3. Interleaving actions. When only considering one agent performing actions a_1, a_2 we obtain a linear sequence

$$\cdot \xrightarrow{a_1} \cdot \xrightarrow{a_2} \cdot$$

However, when two or more agents are involved, there is more than one way of *interleaving* their actions.Indeed, Bob may perform the action b_1 before Alice has performed action a_1, or vice versa. These *execution sequences* will be denoted by words made up of the actions, for example $a_1.b_1.b_2.a_2$ is the execution of the concurrent system in which Alice performs her first task, followed by Bob performing both of his, and then finally Alice completing her second task. Denoting Alice by $A = \{a_1, a_2\}$ and Bob by $B = \{b_1, b_2\}$, we denote the concurrent system they form by $A \mid\mid B$.

When designing a concurrent program, certain execution sequences are undesirable. For example, as described above, errors can occur due to certain actions being inconsistent. One way of verifying the correctness of such a program is to check that the program terminates and does what it is designed to do no matter which order the actions are performed. These different *schedulings* grow exponentially with the size of the program.

1.2.4. Spatial interpretation of concurrency. We sum up all of the possibilities by a spatial interpretation of the situation. Here, the points or nodes correspond to concatenations of actions, while edges correspond to actions. This space will be referred to as the *interleaving graph* in what follows. The concurrent behaviour is expressed by their actions lying along independent axes in the space. In the example of Alice and Bob given above, we obtain:

$$\begin{tikzcd}
\cdot \arrow[r, "a_1"] & \cdot \arrow[r, "a_2"] & \cdot \\
\cdot \arrow[u, "b_2"] \arrow[r, "a_1" description] & \cdot \arrow[u, "b_2" description] \arrow[r, "a_2" description] & \cdot \arrow[u, "b_2"'] \\
\cdot \arrow[u, "b_1"] \arrow[r, "a_1"'] & \cdot \arrow[u, "b_1" description] \arrow[r, "a_2"'] & \cdot \arrow[u, "b_1"']
\end{tikzcd}$$

When moving through the space, we are limited by the irreversibility of time; we must always move either upwards or to the right. The paths through the space that respect this direction correspond to different execution sequences of the set of actions $\{a_1, a_2, b_2, b_2\}$. In this way, this space sums up the possible schedulings associated to the system.

1.3. Execution properties

1.3.1. Confluences. As outlined above, branchings represent a conflict in our system corresponding to a choice in the process of calculation. This indeterministic behaviour is undesirable, since when we calculate we expect a unique answer.

As we have a notion of conflict in calculation, we also have the dual notion of resolution. Given elements x_1, x_2 and x', a *confluence* is a situation of the form

$$\begin{tikzcd}
x_1 \arrow[dr, "*"'] & & x_2 \arrow[dl, "*"] \\
& x'. &
\end{tikzcd}$$

We say that x' is the *target* of the confluence. Note that this is the opposite diagram, in the sense of categorical duality, to the diagram representing a branching. This duality carries over to the interpretation of branchings as conflicts: confluences are situations representing resolutions in the system, in the sense that from two distinct elements, a common (intermediate) answer may be produced.

1.3.2. Resolving conflicts. This becomes more evident when combined with the notion of branching. A branching (resp. local branching) is *confluent* when there exists a confluence *completing* it, as represented in the following diagram:

$$\begin{tikzcd}
& x \arrow[dl, "*"'] \arrow[dr, "*"] & & & & x \arrow[dl] \arrow[dr] & \\
x_1 \arrow[dr, "*"'] & & x_2 \arrow[dl, "*"] & \text{(resp.} & x_1 \arrow[dr, "*"'] & & x_2 \;) \arrow[dl, "*"] \\
& x'. & & & & x'. &
\end{tikzcd}$$

Such diagrams are known as *confluence diagrams.* A conflict appears at the element x, and is resolved to produce, from the distinct answers x_1 and x_2, a single output x'.

1.3.3. Confluence and consistency. The notion of confluence of a branching brings us to the first *consistency properties* of a system of calculation. A system $(X, \to)$ is *confluent* (resp. *locally confluent*) if every branching (resp. local branching) is completed by a confluence. We may also choose to call these properties *directed consistency* (resp. *local directed consistency*) since they express a (directed) consistency or compatibility of the rules amongst themselves.

A zig-zag sequence is *confluent* when there exist directed sequences reducing x and y to a common element z, as illustrated below:

$$\begin{tikzcd}
& x_1 \arrow[dl, "*"'] \arrow[dr, "*"] & & {} \arrow[dl, "*"] & \cdots & {} \arrow[dr, "*"] & & x_k \arrow[dl, "*"'] \arrow[dr, "*"] & \\
x \arrow[drrrr, bend right, "*"'] & & x_2 & & \cdots & & x_{k-1} & & y. \arrow[dllll, bend left, "*"] \\
& & & & z & & & &
\end{tikzcd}$$

An abstract system of calculation in which every zig-zag sequence is confluent is said to have the *Church-Rosser property.* We also say that such an abstract system of calculation is *consistent.* Indeed, this shows a strong compatibility between the reduction rules and the equivalence which is derived from them in the sense that whenever there is an undirected path connecting two elements, there are directed sequences reducing them to a common answer.

1.3.4. Consequences for formal calculation. While confluence, *i.e.* directed consistency, expresses a compatibility of the directed system with itself, the Church-Rosser property expresses a compatibility of the directed system with the undirected equality it generates. In particular, it means that any equivalence can be constructively computed using the directed calculatory steps. This in turn means that the system is deterministic. Indeed, it assures that if a “final” answer can be reached, it is unique.

In terms of automatising calculations, this means that we can freely generate terms and then use the system of calculation to reduce any term to a unique answer. The calculations can thus be performed by a machine.

1.3.5. The Church-Rosser theorem. A classical result in rewriting theory, the *Church-Rosser theorem* [22], states that confluence of a system of calculation is equivalent to it having the Church-Rosser property. In other words, directed consistency of a system assures that it is consistent.

1.3.6. Termination. As described above, confluence properties provide checks for deterministic behaviour of a system of calculation. However, for such a system of calculation to be useful, we must eventually reach an element which is the unique output of the computation. This is formalised using the notions of normal form and termination.

A *normal form* is an element z which cannot be reduced by any reduction rule in the system. More precisely, any element x such that $x \overset{*}{\to} z$ must be syntactically equal to z, *i.e.* $x = z$ in X. Given some element $x \in X$, a *normal form of* x is an element x' such that $x \overset{*}{\to} x'$ and x' is a normal form. When x has only one normal form, we will denote it by $\hat{x}$.

As mentioned above, confluence of the system $(X, \to)$ ensures unicity of normal forms. Existence of normal forms, on the other hand, is ensured by the property of termination. The system $(X, \to)$ *terminates* or is *Noetherian* if there are no infinite reduction sequences. This means that there is no sequence $(x_i)_{i\in\mathbb{N}}$ of syntactically distinct elements of X such that $x_i \to x_{i+1}$ for all $i \in \mathbb{N}$. In a terminating system of calculation, a normal form may be reached from any element.

1.3.7. Noetherian induction. Termination, or Noethericity, not only provides an existence criterion for normal forms, but also gives access to a useful mathematical tool, namely Noetherian induction. Rather than using well-founded induction on the length of a reduction sequence, we may reason by the *distance* of a given element x from a normal form. This is in fact the dual of well-founded induction: instead of proving a property for minimal elements and successors, we prove a property for maximal elements and predecessors.

1.3.8. Newman's lemma. In particular, termination and local consistency imply directed consistency, *i.e.* confluence, from local directed consistency, *i.e.* local confluence. This result, *Newman's lemma* [92] states that given a locally confluent and terminating system of calculation, we may deduce (global) confluence thereof. This is practically useful, as it reduces the consistency check to local branchings.

1.3.9. Consistency theorem. A recurring theme in the study of calculation is transporting local properties to global properties using the structure of the considered system. Combining the two previous results, we obtain just such a local-to-global theorem, which we call the *abstract consistency theorem* in this thesis. This is on the one hand to underline the use of this abstract study of calculation in practice, but also to compare this process to coherence theorems, central to this thesis and outlined in the following section.

The theorem states that given a terminating system of calculation, it suffices to verify consistency locally and directedly in order to conclude consistency of the entire system; it is a direct consequence of Newman's lemma and the Church-Rosser theorem. This combination of these classical theorems will be reflected in the proofs of coherence in calculation.

1.3.10. Consistency in concurrency. When considering concurrent systems, the notion of consistency is also related to confluence-like shapes. Indeed, in this context, consistency should rather express that given two interleavings of actions a and b,

$$\cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot \qquad \text{and} \qquad \cdot \xrightarrow{b} \cdot \xrightarrow{a} \cdot$$

that the sequences $a.b$ and $b.a$ produce the same outcome. In this case, we say the actions are *independent*. A simple example is given by the allocations $x = 4$ and $y = 1$. Since these actions affect different parts of the memory, they can be executed in any order without causing an issue. This leads to the notion of synchronisation: the sequences $a.b$ and $b.a$ are *synchronized* when they are independent. Formally, this means equipping interleaving paths in the state-space with an equivalence relation $\sim$. The situation $a.b \sim b.a$ is represented by the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "a"] & \cdot \\
\cdot \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, phantom, "\sim"] & \cdot, \arrow[u, "b"']
\end{tikzcd}$$

An interleaving graph equipped with such an equivalence relation is called an *asynchronous graph.*

This local consistency can be extended to paths of arbitrary length. Indeed, consider the situation in which Alice is performing the sequence of actions $\{a_1, a_2\}$ interleaved with Bob performing actions $\{b_1, b_2\}$. If each of Alice's actions are independent to those of Bob, we may conclude that any interleaving of their actions are pairwise consistent, see the diagram below.

$$\begin{tikzcd}
\cdot \arrow[r, "a_1"] & \cdot \arrow[r, "a_2"] & \cdot \\
\cdot \arrow[u, "b_2"] \arrow[r, "a_1" description] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "b_2" description] \arrow[r, "a_2" description] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "b_2"'] \\
\cdot \arrow[u, "b_1"] \arrow[r, "a_1"'] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "b_1" description] \arrow[r, "a_2"'] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "b_1"']
\end{tikzcd}$$

For verification of concurrent programs, it now suffices to check only one execution sequence in each equivalence class under the extension of $\sim$ to paths in the interleaving space to assure that the program is correct. A complete study of asynchronous graphs and verification of concurrent programs can be found in [43].

1.3.11. Mutexes. To avoid that dependent actions take place simultaneously, we employ *mutexes*, a term derived from *mutual exclusion*. These can be thought of as resources which only one agent may possess at a given time. Given a mutex m, an agent can either *lock* or *release* the mutex; these actions are denoted by p_m and v_m, respectively[1]. For example, for actions a and b, we can consider the concurrent system $\{p_m, a, v_m\} \mid\mid \{p_m, b, v_m\}$. This situation is described in the diagram below:

$$\begin{tikzcd}
\cdot \arrow[r, "p_m"] & \cdot \arrow[r, "a"] & \cdot \arrow[r, "v_m"] & \cdot \\
\cdot \arrow[u, "v_m"] \arrow[r, "p_m" description] & \cdot \arrow[u, "v_m" description] \arrow[r, "a" description] & \cdot \arrow[u, "v_m" description] \arrow[r, "v_m" description] & \cdot \arrow[u, "v_m"'] \\
\cdot \arrow[u, "b"] \arrow[r, "p_m" description] & \cdot \arrow[u, "b" description] \arrow[r, "a" description] & \cdot \arrow[u, "b" description] \arrow[r, "v_m" description] & \cdot \arrow[u, "b"'] \\
\cdot \arrow[u, "p_m"] \arrow[r, "p_m"'] & \cdot \arrow[u, "p_m" description] \arrow[r, "a"'] & \cdot \arrow[u, "p_m" description] \arrow[r, "v_m"'] & \cdot \arrow[u, "p_m"']
\end{tikzcd}$$

There are now only two execution sequences, the two outer edges of the above square, and they are *not* synchronized. Indeed, whichever agent takes the mutex m first must complete their task, *i.e.* a or b, and release the mutex before the other agent may begin. We again refer to the book [43] for a more complete treatment.

1.3.12. Deadlocks. While mutexes can be used to differentiate execution sequences producing different results, they can cause problems for termination of the program. Indeed, given mutexes m and μ, consider the concurrent system in which Alice and Bob are given the tasks

$$A = \{p_m, p_\mu, v_m, v_\mu\} \mid\mid \{p_\mu, p_m, v_\mu, v_m\} = B,$$

i.e. only lock and release actions. The asynchronous graph associated to this system is given below:

---

[1] In Dutch, *preneer* and *vrijen* mean take and release, respectively.

$$\begin{tikzcd}
\cdot \arrow[r, "p_m"] & \cdot \arrow[r, "p_\mu"] & \cdot \arrow[r, "v_\mu"] & \cdot \arrow[r, "v_m"] & \cdot \\
\cdot \arrow[u, "v_\mu"] \arrow[r, "p_m"] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "v_\mu"] \arrow[r, "p_\mu"] & \cdot \arrow[u, "v_\mu"] \arrow[r, "v_\mu"] & \square \arrow[u, "v_\mu"] \arrow[r, "v_m"] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "v_\mu"'] \\
\cdot \arrow[u, "v_m"] \arrow[r, "p_m"] & \cdot \arrow[u, "v_m"] \arrow[r, "p_\mu"] & \cdot \arrow[u, "v_m"] \arrow[r, "v_\mu"] & \cdot \arrow[u, "v_m"] \arrow[r, "v_m"] & \cdot \arrow[u, "v_m"'] \\
\cdot \arrow[u, "p_m"] \arrow[r, "p_m"] & \bigcirc \arrow[u, "p_m"] \arrow[r, "p_\mu"] & \cdot \arrow[u, "p_m"] \arrow[r, "v_\mu"] & \cdot \arrow[u, "v_m"] \arrow[r, "p_m"] & \cdot \arrow[u, "p_m"'] \\
\cdot \arrow[u, "p_\mu"] \arrow[r, "p_m"'] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "p_\mu"] \arrow[r, "p_\mu"'] & \cdot \arrow[u, "p_\mu"] \arrow[r, "v_\mu"'] & \cdot \arrow[u, "p_\mu"] \arrow[r, "v_m"'] \arrow[ur, phantom, "\sim"] & \cdot \arrow[u, "p_\mu"']
\end{tikzcd}$$

The execution sequences $p_m.p_\mu$ and $p_\mu.p_m$ are independent since the mutexes are distinct. However, if either of those sequences is executed, the process stops. Indeed, Alice is waiting to receive μ in order to release m, while Bob is waiting to receive m in order to release μ. This situation is known as *deadlock*, and corresponds to the circled point in the above diagram. These situations are clearly undesirable: the program described by the above asynchronous graph contains schedulings which will lead to non-termination. Dually, the square point is *unreachable*.

1.4. Coherent calculation

1.4.1. Comparing paths. Consistency in calculation provides a criterion by which to measure the legitimacy of a system of calculation with respect to a notion of equivalence on the considered elements. The consistency theorem for abstract systems of calculation essentially states that under the hypothesis of termination and local confluence, *any* sequence of calculatory steps will result in the *same* unique answer.

Spatially interpreted, this means that we can "slide" the nodes in the graph representing the system along the directed paths corresponding to reduction sequences while contracting the edges along the given direction until we are left with only one point. We illustrate this with the simple example of two calculatory steps:

$$\begin{tikzcd}
& 2+5+4 \arrow[dl] \arrow[dr] & \\
7+4 \arrow[dr] & & 2+9 \arrow[dl] \\
& 11 &
\end{tikzcd}
\qquad
\begin{tikzcd}
7+4 \arrow[dr] & & 2+9 \arrow[dl] \\
& 11. &
\end{tikzcd}
\qquad
11.$$

Spatially, consistency may therefore be interpreted as a *directed contractibility* with respect to points, *i.e.* objects of dimension zero. This is the calculatory parable of the notion of equivalence class. Indeed, the properties of an equivalence relation allow one to consider each equivalence class as an element in a consistent way, while the property of *convergence*, *i.e.* termination and confluence, of a calculatory system allow us to contract the equivalence classes along directed paths to the chosen representative, *i.e.* the unique normal form in each class.

However, while contracting in this way, we break certain topological features of the calculation space. Indeed, in the above example, the “hole” formed by the confluence diagram on the left is broken in the middle diagram. This is therefore not a well defined notion of topological contractibility. This topological inconsistency is related to the freeness of the system of calculation in a way we make clear in the following paragraphs.

1.4.2. Free structures. In the running example of computing sums, expressions such as $2+5+4$ are *freely generated* by natural numbers $\mathbb{N}$. Indeed, the expression $2+5+4$ can be thought of as the *word* 2.5.4. We then equip the set $\mathcal{W}(\mathbb{N})$ of such expressions or words with the usual notion of equivalence $\equiv$ for addition of natural numbers. The quotient of $\mathcal{W}(\mathbb{N})$ by $\equiv$ is isomorphic to the set $\mathbb{N}$ of natural numbers: the equivalence class associated to some number n being represented by the singleton word n.

The notion of calculation comes in when we also equip the set $\mathcal{W}(\mathbb{N})$ with a sub-system of directed rules, expressing calculation toward the representative of each class. These rules *freely generate* reduction sequences. However, certain distinct reduction sequences calculate the *same* output from the same input. In this sense, we may consider that the rules themselves are subject to some notion of equivalence.

1.4.3. Higher witnesses. The advantage of using free structures is that they are in some sense automatisable. Given a set of letters, in our case the set of natural numbers, we can automatise the construction of the words, or expressions, in the set $\mathcal{W}(\mathbb{N})$. However, when the rules that we use are not freely generated, we must specify all of them, a time-consuming and non-automatisable process.

For this reason, the goal is to build, step by step, higher relations from the one dimensional information given by the system, which are themselves subject to even higher relations. In this way, the entire system of calculation is freely generated.

1.4.4. Higher formalisation. The 1-dimensional properties and mechanisms of calculation can be expressed in relational or Kleene algebraic settings. These algebraic structures have been extensively formalised in interactive theorem provers like Isabelle[2] and Coq[3]. These formalisations have already found applications in program verification,

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[2] See the Isabelle theories Kleene algebra and Kleene algebra with domain.

[3] See the Coq library on relational and Kleene algebras.

see [2, 45] and [118] for applications of Isabelle/HOL to program verification[4], the latter being in the case of hybrid systems, and [94] for applications of Coq. Furthermore, classical rewriting theorems, originally expressed in relational algebra, have been generalised in Kleene algebraic structures [28, 109].

However, the higher mechanisms of unwinding a system of calculation to obtain a truly free presentation, described in the following paragraphs, have not yet been formalised. In order to exploit the mathematical tools offered by this higher analysis of calculatory systems, we require a structure with a simple algebraic signature that captures this process. Such structures, introduced later in this thesis document, provide first steps toward a formalisation of the mechanisms by which a system calculation is replaced by a free higher system of calculation. Since free structures are those which can be handled by a machine, this leads towards a wholly free, automatised paradigm of calculation.

1.4.5. Spheres and equivalences. When two reductions have the same source and target, we say that they are *parallel*, and that together, they form a *sphere*. Since spheres represent pairs of reduction sequences calculating the same thing, we would like them to be equivalent. We again denote this equivalence by $\equiv$, but now it applies to reduction sequences, rather than elements. Given the (labelled) rules

$$a_1 : 2+5+4 \to 7+4, \qquad a_2 : 7+4 \to 11,$$
$$b_1 : 2+5+4 \to 2+9 \qquad b_2 : 2+9 \to 11,$$

we set $a_1 a_2 \equiv b_1 b_2$ to indicate that these reduction sequences perform the same calculation in two different ways. This is graphically indicated as below:

$$\begin{tikzcd}
 & 2+5+4 \arrow[dl, "a_1"'] \arrow[dr, "b_1"] & \\
7+4 \arrow[dr, "a_2"'] & \equiv & 2+9 \arrow[dl, "b_2"] \\
 & 11 &
\end{tikzcd}$$

1.4.6. Coherence. If we consider *all* of the spheres in a system of calculation $(X, \to)$, we can provide witnesses stating that *all* of the reduction sequences are equivalent. However, this corresponds again to specifying *all* of the rules relating paths, a lengthy process in general. As before, we would like to start with some *generating* higher witnesses, and "glue" them together in order to deduce new equivalences.

Given some set Γ of higher (globular) witnesses, which can be thought of as *tiles* filling the space between parallel reduction sequences, we may ask ourselves the following question:

> Given any two zig-zags, *i.e.* equivalences, can we fill the sphere created between them with the tiles in Γ?

---

[4] see also the associated Isabelle theory.

When the answer to this question is yes, we say that $(X, \to)$ is *coherent* with respect to the tiles in Γ. Coherence essentially means that we have a well-defined notion of equivalence on the reduction sequences in the given calculatory system. Recall from the above that this notion of equivalence is semantically interpreted in the following way: two zig-zags which represent the same equality but composed of different calculatory steps are equivalent. Coherence is important in categorical algebra, in which structure at a given level may be encoded by higher cells. An example of this is the coherence of strict monoidal categories, as demonstrated by MacLane [88].

1.4.7. A higher system of calculation. We may now consider a system of calculation underlying this higher notion equivalence. Indeed, by choosing a *direction* for the elements of Γ, we obtain a system of calculation on the set of paths.

$$\begin{tikzcd}
 & 2+5+4 \arrow[ld, "a_1"'] \arrow[rd, "b_1"] & \\
7+4 \arrow[rd, "a_2"'] & \Rightarrow & 2+9 \arrow[ld, "b_2"] \\
 & 11 &
\end{tikzcd}$$

Adding labels to these higher calculatory steps, we may now ask the same question of this new, higher-dimensional system of calculation, and so on. When we can “unfold” all of these higher systems, we obtain a higher structure which is free in all dimensions.

1.4.8. Spatial interpretation of coherence. As described in Section 1.4.1, consistency can be thought of as a notion of zero-dimensional contractibility along directed paths. Coherence also has a spatial interpretation. When a confluence diagram is filled with a tile, as depicted below, we can contract the paths along this two dimensional space into one, and then shrink the resultant, single path, along the indicated direction toward the target of the confluence. As opposed to the case of consistency, we do not break any two-dimensional holes in this operation.

$$\begin{tikzcd}
 & x \arrow[ld, "l_1"'] \arrow[rd, "r_1"] & & & x \arrow[dd] & & \\
x_1 \arrow[rd, "l_2"'] & \equiv & x_2 \arrow[ld, "r_2"] & & & & \\
 & x' & & & x' & & x'
\end{tikzcd}$$

However, once all of the branchings are filled with two-dimensional tiles, three-fold branchings will, in general, cause three-dimensional holes to appear. In the below diagrams, we consider a three-fold branching (l_1, c_1, r_1) which are pairwise resolved by confluences. The borders are identified, resulting in a 2-sphere:

$$\begin{tikzcd}
& & x \arrow[dll, "l_1"'] \arrow[d, "c_1"] \arrow[drr, "r_1"] & & & & & x \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & \\
x_1 \arrow[drr, "l_2"'] & \equiv & x_3 \arrow[d, "c_2"] & \equiv & x_2 \arrow[dll, "r_2"] & & x_1 \arrow[dr, "l_2"'] & \equiv & x_2 \arrow[dl, "r_2"] \\
& & x' & & & & & x' &
\end{tikzcd}$$

A coherent structure is therefore truly contractible: *every* "hole" of *any* dimension has been filled, allowing the whole system to be retracted in a topologically sound manner. Formally, such higher dimensional systems of calculation, in their spatial interpretation, may be modelled by *polygraphs* [11], first called *computads* [106, 107].

1.4.9. Spatial interpretation of higher systems. Just as the equivalence tiles may be thought of as two-dimensional tiles filling two-dimensional holes, we have notions of n-dimensional tiles filling n-dimensional holes. When we take a directional perspective on these higher rules, we may think of them as homotopies, *i.e.* paths between paths. Topologically, a system which is coherent in all dimensions is equipped with contracting homotopies, starting from the highest (possibly infinite) dimension, contracting it to the one below, which is then contracted again, and so on until we reach a zero-dimensional structure.

$$\begin{tikzcd}
& x \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & & & x \arrow[dr, "r_1"] & & & & & & \\
x_1 \arrow[dr, "l_2"'] & \Rightarrow & x_2 \arrow[dl, "r_2"] & & & x_2 \arrow[dl, "r_2"] & & & x_2 \arrow[dl, "r_2"] & & \\
& x' & & & x' & & & x' & & & x'
\end{tikzcd}$$

1.4.10. A coherence theorem. As in the case of consistency, we may use the calculatory properties of the one dimensional structure to go from local coherence properties to global coherence. This follows the same proof structure as the consistency theorem for locally confluent and terminating abstract systems of calculation.

Let $(X, \to)$ be an abstract system of calculation, and set Γ of higher relations on the reduction sequences thereof, directed or not. If any two zig-zags may be connected by glueing together elements of Γ, then Γ is called a *homotopy basis* for $(X, \to)$. This in particular means that $(X, \to)$ is coherent with respect to Γ.

When $(X, \to)$ is locally confluent, we specify a higher relation constructed using this property. A *family of generating confluences* for $(X, \to)$ is an equivalence on reduction sequences in the system consisting of a tile $\equiv_{r,l}$ for each local branching (r, l). Given labelled rewriting steps r and l constituting a branching, we choose a confluence (r', l')

completing this branching and impose $rr' \equiv ll'$, as illustrated below:

$$\begin{tikzcd}
 & x \arrow[dl, "r"'] \arrow[dr] & \\
x_1 \arrow[dr, "*"'] & \equiv_{r,l} & x_2 \arrow[dl, "*"] \\
 & x'. &
\end{tikzcd}$$

The coherence theorem states that a family of generating confluences constitutes a homotopy basis, first proved by Squier in [104] to study coherence problems for monoids. In this sense, we see again the schema of the consistency theorem, namely that from a local property, the family of generating confluences, we obtain a global property, namely coherence. The key difference is the higher witnesses provide a means of propagating the constructive approach of rewriting to higher dimensions.

1.4.11. Strategies. If we choose a direction for the tiles $\equiv_{r,l}$ associated to local branchings, we may consider it as a system of calculation on the reduction sequences of $(X, \rightarrow)$. This provides a new dimension of rewriting. The one-dimensional property of local confluence and termination allows us to conclude that this system of calculation is also coherent with respect to a yet higher system of relations. This process can be continued, proving consistency and then coherence of each layer, and this from only the information provided by the original one-dimensional system [65].

This propagation of convergence at each dimension of rewriting is encoded by the notion of *strategy* [67]. While a convergent system of calculation provides a notion of normal form, the notion of strategy provides a higher dimensional analog. Indeed, after “choosing” base-points for zero-dimensional contractibility in the guise of normal forms, we may also specify a base-point for reduction sequences, and then zig-zags. These then serve as normal forms for the two-dimensional system of calculation, providing a notion of one-dimensional contractibility, and so on ad infinitum.

1.4.12. Formalising coherence. As described in Section 1.4.4 the mathematics of consistency in formal calculation have found an expression in simple algebraic structures, first in relation algebras and then in the more general setting of Kleene algebras. This leads to the following question:

> Is there an algebraic setting in which all dimensions of rewriting, in particular higher paving mechanisms, are captured and which generalises the one-dimensional setting?

In Part I of this thesis, we tackle this question by introducing a notion of *higher Kleene algebra* in which higher consistency theorems and the abstract coherence theorem are captured.

1.5. An algebraico-topological approach to concurrency

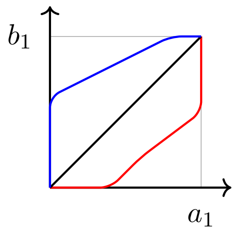
1.5.1. True concurrency. In the modelisation of concurrent systems provided by asynchronous graphs, the actions performed by each agent are seen to take place instantaneously: several actions cannot take place at the same time. In this sense, actions are atomic, that is no time passes while the action is taking place. A richer description of such a system allows for Alice and Bob to both be in the process of completing an action *simultaneously*. This provides a finer description of concurrency, since it is exactly the indeterminacy of the time it takes for each task to be completed which makes different execution sequences possible.

Interpreting this topologically means allowing the model described above to allow for paths to occupy the space between the sequences represented in the interleaving model. For example, in the latter, the sequences “Alice does a_1 followed by Bob does b_1” and “Bob does b_1 followed by Alice does a_1”, is represented by

$$\begin{tikzcd}
\cdot \arrow[r, "a_1"] & \cdot \\
\cdot \arrow[u, "b_1"] \arrow[r, "a_1"'] & \cdot, \arrow[u, "b_1"']
\end{tikzcd}$$

in which only execution sequences following the edges labelled a_1 and b_1 are allowed.

1.5.2. Spatial interpretation of true concurrency. In the *true concurrency* point of view, see [43], we allow for Alice to do a_1 *while* Bob does b_1, allowing for richer range of possibilities concerning execution sequences. These more aptly describe the different situations which may arise by allowing for tasks to take time. A few such situations are represented in the diagram below in the case of the toy example of Alice and Bob performing actions a_1 and b_1 concurrently.

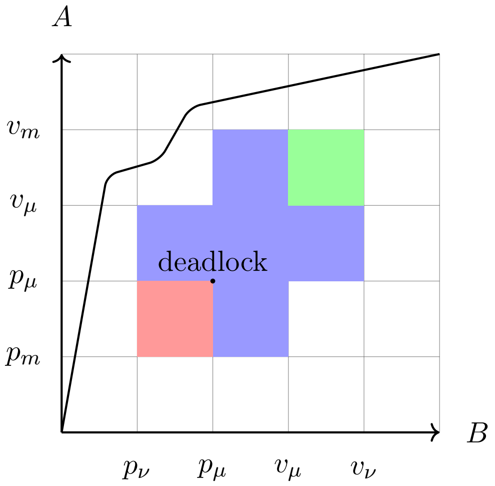


(1.5.3)

Note that here we represent the action a_1 (resp. b_1) as the interval preceding the label, that is, the action is completed once we reach the vertical (resp. horizontal) gray line. The blue path represents an execution in which Alice starts a_1 before Bob starts b_1 and also finishes before Bob, whereas the red path expresses the opposite situation. The black diagonal path represents an execution in which Alice and Bob both start and finish at the same time.

1.5.4. Directed spaces. There are of course still rules about direction in this space: we must also stick to paths that always point toward the top-right. Generalising this idea, we introduce the notion of *directed space*. These are spaces in which certain paths, the *directed paths*, are designated as those providing directional information, see [43, 61]. A partially ordered space equipped with its set of monotone maps from the unit interval with its usual order is an example of such a directed space. Directed topology was originally introduced as a model, and a tool, for studying and classifying concurrent systems in computer science [57, 95]. These can be combinatorially described not by asynchronous graphs, but by the richer notion of *cubical complexes*, also called higher dimensional automata, see [95].

Considering the example of the concurrent system with two mutexes described in Section 1.3.12, we have the following true concurrency model as a pospace: we consider a square in $\mathbb{R}^2$ equipped with the pointwise partial order, *i.e.* $(x, y) \leq (x', y') \iff x \leq x'$ and $y \leq y'$, with a "plus" shaped hole, corresponding to the blue area in the following diagram:



(1.5.5)

Valid executions of this system correspond to paths which are non-decreasing for the partial order, for example the black curve. This provides a richer description than the interleaving model described in Sections 1.2.4 and 1.3.10, an example being that any path which enters the red square is *doomed* to end in deadlock.

Formally, a directed space $\mathcal{X}$ consists of a topological space X and a set dX of directed paths. The set dX is stable under concatenation, since two execution sequences should be able, under certain conditions, to be put together to form a longer sequence, and under reparametrisation, since the "speed" of an execution sequence is not important, but rather the points through which the corresponding path in the space passes since this determines the scheduling of actions. Lastly, we ask that it contains all of the stationary paths, *i.e.* those which send the entire interval to a point. A formal definition is given in Section 11.6.1.

1.5.6. Higher spatial analysis. Not only does this allow for a richer variety of execution sequences, but introducing a continuous topological model allows us to interpret synchronisation of paths topologically. Indeed, in this optic, paths are equivalent when there are no holes in the space separating them.

So far we have only seen systems involving two agents. However, in general, concurrent systems can consist of many agents, or, as in the case of directed spaces, be abstracted from the notion of agents altogether. For this reason, a study of holes in the space of *any* dimension becomes important to detect the areas of mutual exclusion. Indeed, a hole in the space of dimension n corresponds to an area of mutual exclusion involving n agents.

However, a system involving n concurrently acting agents may have holes of dimension $k < n$. In practice, this corresponds to generalised mutexes. Instead of only being allowed to be held by *one* agent at a time, these resources are assigned an arity. More precisely, a *resource* (r, k_r) can be held by k_r agents at a time. This can produce holes of dimension k_r.

Tracking the use of these objects becomes very difficult when there are many agents, and many resources of various arities [43]. For this reason, classifying concurrent systems with algebraic invariants can be useful as criteria for program design and verification. Since we have a continuous topological model of such systems in the context of true concurrency, we draw from tools in algebraic topology.

1.5.7. Algebraic invariants. Homotopy and homology are algebraic invariants of topological spaces. They both measure the existence of holes in the space, by “tethered” (directed higher dimensional) loops in the case of the former, and by “untethered” (directed higher dimensional) loops in the case of the latter. Algebraically, they are distinguished by being groups and modules, respectively. Computationally, homology is a more tractable invariant, morally because of its “unfettered” nature, whereas homotopy provides a finer study of “wrapping properties”, making it harder to calculate. For an excellent account of the basics, we refer the reader to the standard textbook [70].

Since directed paths are, in general *not* loops, it could be that no such paths exist in the space. Furthermore, while homology and homotopy are invariants of usual topological spaces, and hence about the properties of the points of the space, the objects of interest in the study of concurrent systems are the execution sequences, which correspond to the directed paths. Our objects of study are thus no longer zero dimensional points of the underlying space, but one dimensional (directed) paths.

1.5.8. Spaces of paths. For this reason, we study the topological properties of *spaces of directed paths*. As in the case of systems of calculation, we limit our study to spaces of paths which begin and end at the same two points, which will again be referred to as *parallel*. Looking back to the asynchronous graph model of concurrency, we see that such parallel paths correspond to some fixed set of actions having been completed.

Given two points x and y in the space, the set $P_{x,y}$ of parallel directed paths between them is naturally endowed with a topology inherited from the underlying space X. We can thus apply the classical invariants described above to these spaces, and thereby obtain information about the holes between paths, *i.e.* areas of mutual exclusion. However, there are many such spaces of parallel directed paths in $\mathcal{X}$; we would like to also measure how the topological features of these spaces vary as the beginning and end points move.

1.5.9. Directed algebraic invariants. Given a directed path v from y to some third point z, we obtain a continuous map from $P_{x,y}$ into $P_{x,z}$ by post-concatenating elements of $P_{x,y}$ with v. We then also obtain homomorphisms between the corresponding homotopy groups or homology modules. This allows us to detect the appearance of a hole between directed paths. Of course, we can also pre-concatenate with a path u, starting at some point w and ending in x. Without *extending* using pairs of directed paths (u, v), called *extensions*, there is in general no canonical way to obtain continuous maps $P_{x,y} \to P_{w,z}$.

The ensemble of all such spaces and their inclusions by extensions provide a diagram of topological spaces, to which we can apply homotopy and homology functors. This results in a diagram of groups or modules, the homomorphisms of which measure the evolution of topological features as we extend along directed paths. These diagrams are referred to as *natural homotopy* and *natural homology*, see [31], in reference to their being encoded as *natural systems*, a categorical gadget originally used in the cohomology theory of small categories, see [5].

1.5.10. Time-sensitivity. After these invariants were introduced, notably after natural homology was explored in [32], it was pointed out [87] that they did not capture *time-reversal*. Given a directed space $\mathcal{X} = (X, dX)$, we may consider its time-reversal, denoted by $\mathcal{X}^o$. This directed space has the same underlying topological space X, but its set of directed paths dX^o are those of $\mathcal{X}$ reversed, that is

$$dX^o := \{t \mapsto f(1-t) \mid f \in dX\}.$$

It is of course not desirable that a directed space should have an isomorphic invariant to its time-reversal. However, in the original definition of the invariants, this was the case. This leads naturally to the following question:

> How can natural homology and natural homotopy be augmented to distinguish a directed space and its time-reversal?

The answer to this question was found in the notion of *composition pairing*, which relates the concatenation of paths to the homology or homotopy groups of path spaces. Composition pairings are actually part of the structure of lax functors, as described by Porter in [93]. This is explained in Chapter 12 in Part II.

1.5.11. Tractability. In classical algebraic topology, homotopy groups are typically harder to calculate than homology modules, the latter in fact lending themselves to calculation. For this reason, in the context of directed spaces, it is still true that natural homology is more tractable than natural homotopy, but it is nonetheless a large invariant, making it difficult to compute.

Indeed, for each pair of points which are the beginning and end of some directed path, we must calculate the homology module associated to the space. In addition to this, we must calculate the induced maps given by every possible extension in order to obtain the entire natural homology diagram. This quickly becomes too large to be tractable in practice. However, it is exactly the homology maps induced by extensions which provide information about the execution properties of the concurrent program modelled by the considered space.

Persistent homology is another application of classical homology used in topological data analysis, see [19] for an overview thereof. It is widely used in the spatial analysis of data sets, typically given by high dimensional vectors. The idea is to take the discrete set of points representing the considered data and "expand" each point, thereby creating a sequence of larger and larger spaces. Calculating the homology of each of these, and in particular, the maps induced by their inclusions, allows to capture the spatial features of the data while eliminating noise. Importantly, uni-dimensional persistence modules are computationally tractable.

The importance of the induced maps in each of these applications of classical homology leads to the following question:

> Is there a link between natural homology and persistent homology, and if so, can the latter be used to calculate the former more efficiently?

In response, we establish that natural homology is in fact a generalised persistence module, see Section 11.7 in Part II. The directed structure provides uni-dimensional persistence modules by considering extensions along a fixed directed path. Furthermore, we show that this information may be amalgamated and thereby reconstitute the natural homology diagram, making a first step towards computability of this invariant.

CHAPTER 2.
RELATIONAL AND ALGEBRAIC MODELS OF CALCULATION

Now we turn to mathematical models of calculation, first in simple algebraic structures. The first, borrowing concepts from relational algebras, encapsulates the notion of *abstract rewriting systems*. These are transition systems given by a binary relation over a set whose elements are the objects on which calculations are performed. The second consists of a generalisation of relation algebras, namely *Kleene algebras*, first introduced in the study of regular languages and automata by Conway [23] under the name *regular algebras*. These form a Türing complete model of calculation. When augmented with a modal structure, calculatory properties and techniques are captured, see for example [28], or [27] for a survey of modally enriched Kleene algebraic structures.

These algebraic structures capture the one-dimensional properties of calculation, but do not provide the machinery for tackling higher dimensional analysis of the considered system. However, they are formalised in most proof assistants, for example in the proof assistant Isabelle[1] and in the proof assistant Coq[2], allowing verification of one dimensional calculatory properties.

We describe the properties of calculation discussed in Chapter 1, first in the context of abstract rewriting systems in Section 2.1, and then in modal Kleene algebra in Section 2.2.

2.1. ABSTRACT REWRITING SYSTEMS

The classical description of abstract systems of calculation is via relations [4]. While many rewriting proofs have a geometric interpretation, description of abstract rewriting in the relational setting has the advantage of expressing important properties by inclusions of sets. This means on the one hand that proofs can be expressed formally, and provides an algebraic interpretation on the other; binary relations over a fixed set form an algebra. For a more complete account of ARS, see Section 5.1, or consult the standard

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[1] See the Isabelle theories on Kleene algebras and Kleene algebras with domain.

[2] See the Coq library for relational and Kleene algebras.

references [4, 111].

2.1.1. Transitions as relations.

A system of calculation described by a relation is called an *abstract rewriting system* (ARS). Formally, they consist of a set X of *objects* and a binary relation R on X, *i.e.* a subset of the product $X \times X$.

The symbolic statement $(x, y) \in R$ is interpreted as "y can be calculated from x as the result of an elementary calculation", and is referred to as a *step*. We say that x is *rewritten* to y. Due to the directed nature of calculation, we often denote these relations by arrows $\to_R$, or simply $\to$ when no confusion is possible.

As is described formally in Section 5.1, binary relations over a fixed set form an algebra equipped with a notion of *composition* and of *union*. We can express all of the calculatory properties discussed in Chapter 1 such as reduction sequences, (local) confluence, the Church-Rosser property, and termination using these algebraic operations.

2.1.2. Closures, branchings and confluences.

Relational composition is a non-commutative operation akin to a multiplication. It is defined formally in the following way: given binary relations R and S on X, their *composition* $R \circ S$ is the relation defined by

$$(x, z) \in R \circ S \qquad \iff \qquad \exists y \in X,\ (x, y) \in R \,\text{and}\, (y, z) \in S.$$

Given an ARS $\to$, we may therefore define reduction sequences, henceforth referred to also as rewriting sequences, as powers $\to^n$ of the relation $\to$ under this notion of composition. However, $(x, y) \in \to^n$ means that x and y are connected by a sequence of n calculatory steps.

To obtain the relation corresponding to reduction sequences of arbitrary length, we consider the union of all the powers of $\to$:

$$\overset{*}{\to} = \bigcup_{i \in \mathbb{N}} \to^i,$$

where $\to^0$ is interpreted as the neutral element $\Delta = \{(x, x) \mid x \in X\}$ for the composition operation. This is the *reflexive, transitive closure* of $\to$, *i.e.* the smallest reflexive, transitive relation containing $\to$. This relation represents the reduction sequences of the ARS.

We may also consider the *symmetric, reflexive, transitive closure* of $\to$, denoted by $\overset{*}{\leftrightarrow}$. This is the smallest equivalence relation containing $\to$, and hence corresponds to the notion of equivalence underlying the system of calculation represented by the ARS $\to$.

The notion of composition also provides relations representing the branchings and confluences of the ARS. Denoting by $\leftarrow$ the *converse* relation of $\to$, *i.e.* $(y, x) \in \leftarrow \iff (x, y) \in \to$, local branchings correspond to the composition $\leftarrow \circ \to$. Indeed, $(x_1, x_2) \in \leftarrow \circ \to$ means by definition that there exists (at least) an element $x \in X$ such that $x \to x_1$ and $x \to x_1$.

A (global) branching corresponds to a similar situation, but instead of reduction steps, we should have reduction sequences. Branchings thus correspond to the composition $\overset{*}{\leftarrow} \circ \overset{*}{\rightarrow}$.

Dually, confluences are represented by $\overset{*}{\rightarrow} \circ \overset{*}{\leftarrow}$, since given an element (x_1, x_2) of this composite, there must exist some element x' such that $x_1 \overset{*}{\rightarrow} x'$ and $x_2 \overset{*}{\rightarrow} x'$.

2.1.3. Termination.

As outlined in the previous chapter, a system of calculation terminates provided that there exist no infinite reduction sequences. In the relational setting, this is expressed using notions from order theory; after all, a partial order is simply a relation with specific properties.

Given a relation R on a set X, we say that an element x is *R-maximal* or simply *maximal* if for all $y \in X$, $(x, y) \notin R$. Given a (possibly infinite) subset $S \subseteq X$ of X, there is *no* R-maximal element in S provided that the set

$$\diamond_R S := \{y \in X \mid \exists x \in S,\, (x, y) \in R\}$$

contains S. Indeed, if this is the case, we can iterate or loop R-steps on elements of S.

For this reason, we say that an ARS $\rightarrow$ *terminates* if for all $S \subseteq X$, if $S \subseteq \diamond_{\rightarrow} S$, then S must be the empty set. This ensures that all subsets of X have a maximal element. This is the dual property of being *well-founded*, and is known as Noethericity after Emmy Noether. We therefore also refer to terminating ARS as *Noetherian*.

2.1.4. Consistency.

As outlined in the previous chapter, consistency is a central property to a well-constructed system of calculation. In the relational setting, these consistency properties can be expressed as set-inclusions.

The property of local confluence corresponds to the following formal statement of set-inclusion:

$$\leftarrow \ \circ \rightarrow \subseteq \overset{*}{\rightarrow} \circ \overset{*}{\leftarrow} .$$

This is local confluence of the entire system of calculation described by the ARS as described in Section 5.1. In the relational setting, we cannot express confluence of a *specific* local branching, since the relation $\leftarrow \circ \rightarrow$ encodes all such configurations at once, just as $\overset{*}{\rightarrow} \circ \overset{*}{\leftarrow}$ encodes all confluence configurations. This is similar in the case of (global) branchings and zig-zags.

Confluence (resp. the Church-Rosser property) of the ARS $\rightarrow$ are similarly expressed as inclusions, given respectively by

$$\overset{*}{\leftarrow} \circ \overset{*}{\rightarrow} \subseteq \overset{*}{\rightarrow} \circ \overset{*}{\leftarrow}, \qquad\qquad \overset{*}{\leftrightarrow} \subseteq \overset{*}{\rightarrow} \circ \overset{*}{\leftarrow} .$$

The relations representing (local) branchings, zig-zags and confluences can be thought of as certain *shapes* or configurations. As can be seen from the definitions of consistency properties in ARS, we relate "conflict shapes", *i.e.* branchings and zig-zags, to "resolution shapes", *i.e.* confluences. This relation is that of set-inclusion.

It is important to note that the existential and universal quantifications on conflict shapes and resolution shapes, respectively, are encoded in this inclusion. Indeed, the definition of local confluence means that for every local branching with targets x_1 and x_2, that is, $(x_1, x_2) \in \leftarrow \circ \rightarrow$, there exists a confluence with sources x_1 and x_2, *i.e.* $(x_1, x_2) \in \xrightarrow{*} \circ \xleftarrow{*}$.

So while we cannot access the specific conflict and resolution shapes, this deficiency is balanced by encapsulating all resolutions of conflicts at once. The consistency theorem becomes:

5.1.11 **Theorem (Consistency for ARS).** *Let $\rightarrow$ be a locally confluent, terminating ARS. Then $\rightarrow$ is Church-Rosser.*

2.1.5. Pathlessness. The relational approach to calculation has many advantages. One of the main benefits we highlight here, is that the rewriting properties described in the previous paragraphs are defined internally to an algebra with a relatively simple signature, namely the full relation algebra on the considered set. However, we lose information about the rewrite sequences themselves.

Indeed, relational composition forgets the "middle points" and thus identifies distinct rewriting paths. For example, if $(x, y), (x, y') \in R$ and $(y, z), (y', z) \in S$, we lose the distinct reduction sequences $xRySz$ and $xRy'Sz$. The only information $R \circ S$ provides is that x is rewritten to z via *some* path. In this sense, relations provide information about *connectedness* of objects, but forget the *choices* made while rewriting.

Consistency properties are therefore expressible in the relational setting, and the proofs take place internally to the algebra. Coherence properties cannot be expressed, since these depend on comparing reduction sequences.

2.2. MODAL KLEENE ALGEBRA

Relational rewriting provides a formal algebraic structure in which calculatory properties may be expressed and results proved. However, they are not the simplest, or more most abstract, algebraic setting in which rewriting techniques may be expressed. For this, we turn to Kleene algebra. These have their origins in language and automata theorem, originally called *regular algebras* [23], but have recently been extensively used for the formalisation of rewriting and program verification techniques, see [27] for a survey.

A central difficulty in formal mathematics is in balancing readability of specifications and proficient automated proof search. Capturing intuitions while remaining formally rigorous constitutes a first stumbling block, which ideally should result in a setting that provides correct, automated proofs which are readable and even illuminating. A powerful formalisation of abstract rewriting theory may be found in the theory of Kleene algebras. Algebraic abstraction allows for simple proofs in which deduction is replaced by calculation [109]. Proofs in this setting reconstruct intuitive proofs by diagrammatic reasoning, making Kleene algebras a formal setting well suited to capture abstract rewriting results. Modal Kleene algebras (MKAs) formalise abstract rewriting systems (ARS), especially with respect to termination and normalisation properties [28, 109]. In particular, Newman's lemma has been proved internally to modal Kleene algebra, and the Church-Rosser theorem has been formalised in the proof assistant Isabelle[3].In this section we briefly describe these structures and their interpretation as a setting for rewriting proofs.

2.2.1. The algebraic structure.

A Kleene algebra is an algebra with the following signature:

$$(K, +, 0, \cdot, 1, (-)^*),$$

where K is the underlying set; we will abuse notation by denoting the whole structure by K. The operations $+$ and $\cdot$ are called *addition* and *multiplication* respectively. Their neutral elements are 0 and 1, called *zero* and *one*. Multiplication distributes over addition on the left and right, and zero is an annihilator for multiplication. This describes the structure of a *semiring*, see Section 6.1.

However, we impose that the addition operation is idempotent, *i.e.* $x + x = x$ for all $x \in K$. This allows an interpretation of the addition as a join-like operation, as well as equipping K with an ordering defined by

$$x \leq y \qquad \Longleftrightarrow \qquad x + y = y.$$

Multiplication and addition are monotone with respect to this order.

The structure of idempotent semiring is augmented with a function $(-)^* : K \to K$ called the *Kleene star*. This map satisfies, for all $x, y, z \in K$,

- **i)** (*unfold axioms*) $1 + x \cdot x^* \leq x^*$ and $1 + x^* \cdot x \leq x^*$,
- **ii)** (*induction axioms*) $z + x \cdot y \leq y \Rightarrow x^* \cdot z \leq y$ and $z + y \cdot x \leq y \Rightarrow z \cdot x^* \leq y$.

These axioms model iterative composition of an element of the algebra. In contrast to the relational setting, this is not defined as a supremum over powers. Rather, it is modelled as a least fix-point. Indeed, consider the following mappings:

$$a \mapsto 1 + xa \qquad \text{and} \qquad a \mapsto 1 + ax.$$

The first axioms state that x^* is a least fixed point of this function. They are called the *unfold axioms* since they express that an iteration of x may be unfolded into either doing nothing, the 1, or doing an x step followed by an iteration of x steps, or vice versa.

The second axioms state that $x^* \cdot z$ and $z \cdot x^*$ are, respectively, least pre-fixed points of the monotonic functions

$$a \mapsto z + a \cdot y \qquad \text{and} \qquad a \mapsto z + y \cdot a.$$

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[3] This theorem may be found in the Kleene algebra Isabelle theory.

They are called the *induction axioms* because they express the principle of well-founded induction based on an abstract notion of length, or size of an iteration. For example, the second induction axiom can be interpreted as follows: the goal of the induction is to prove that, initialising at z, any number of x-steps is under the element y, *i.e.* $z \cdot x^* \leq y$. To prove this, it suffices to show that

- z holds, *i.e.* $z \leq y$.
- a single x-step followed by y holds, *i.e.* $x \cdot y \leq y$.

From these two statements, we obtain by idempotence of addition that $z + x \cdot y \leq y$, allowing us to deduce the claim we set out to prove.

The Kleene star is algebraically characterised, allowing us to perform simple, induction-free proofs, that are well suited for automation. Since relational algebra satisfy continuity of multiplication, *i.e.* composition, over addition, *i.e.* set-union, Kleene algebraic proofs are more general.

2.2.2. Modalities. The structure of Kleene algebra may be further augmented by notions of *domain* and *codomain*. These are maps

$$d : K \to K \qquad \text{and} \qquad r : K \to K,$$

satisfying a number of axioms. The domain operation, for example, satisfies the following five axioms:

$$x \leq d(x)x, \qquad d(xy) = d(xd(y)), \qquad d(x) \leq 1,$$
$$d(0) = 0, \qquad d(x+y) = d(x) + d(y).$$

Those for codomain are similar: the order of multiplication is reversed. These first two axioms encode, respectively, that restricting an element x on the left to its domain contains x, and that domain is *local* in the sense that the domain of a product xy depends only on the domain of x when restricted on the right to the domain of y, see below. The first axiom may strengthened to an equality by considering the third and using monotonicity of multiplication. The other axioms express that elements in the image of d are *sub-identities*, *i.e.* are below 1, and that d is a morphism of sup-semilattices.

The domain axioms impose that the set of fix-points of d coincides with its image:

$$K_d := \{x \in K \mid d(x) = x\} = d(K).$$

The set K_d is called the *domain algebra* of K. This is a sub-algebra of K in which the multiplication is commutative and idempotent, endowing it with the structure of a lattice. Elements of this subalgebra are called *domain elements*.

Elements in K act on the domain algebra via modal operators. Indeed, the domain and codomain axioms imply that

$$\begin{aligned} |x\rangle : K_d &\longrightarrow K_d \\ p &\longmapsto d(x \cdot p) \end{aligned} \qquad\qquad \begin{aligned} \langle x| : K_d &\longrightarrow K_d \\ p &\longmapsto r(p \cdot x), \end{aligned}$$

called *forward* and *backward diamond operators* associated to x, respectively, are hemimorphisms of the lattice K_d, *i.e.* commute with binary meets and joins, send 0 to 0 and commute with suprema. They satisfy module-like laws, e.g.$|x\rangle|y\rangle = |xy\rangle$, and coincide with the diamond operators associated to relations in the standard Kripke semantics[51].

Diamond operators represent reachability relations, *i.e.* tell us which sets are "connected" by which reductions. For example, the inequality $\langle x|(p) \leq q$ is interpreted as the statement " starting in p, x-steps take us into q".

We also consider a *converse* operation in the form of an involution $(\overline{\ })$: $K \to K$ satisfying a number of axioms, see Section 6.1.17.

2.2.3. An algebra of transition systems. We now turn to the intuitions underlying the use of this algebraic structures as a means of describing systems of calculation. So far, we have discussed abstract systems of calculation and rewriting systems. Both consist in an underlying set X and a number of reductions relating elements thereof. Here, we have an abstract representation of the power-set of X as well as *many* transition or reduction relations on elements of X.

Indeed, the domain algebra of a Kleene algebra K may be thought of as a lattice of subsets of a set X due to its lattice structure. However, elements of K_d may be thought of not only as subsets of X but also *stationary* reductions on X, *i.e.* reductions which reduce elements to themselves. The entire set of objects is represented by the element 1, the top of the lattice of domain elements.

Other elements of K are interpreted as collections of (non-stationary) transitions on elements of X. In this sense, an element $x \in K$ can be thought of as a relation, but may have other interpretations, such as a set of paths. A Kleene algebra is thus an algebra of transition systems. Multiplication represents concatenation, *i.e.* $x \cdot y$ is the element which follows x-steps by y-steps. Multiplying an element $x \in K$ on the left (resp. right) by a domain element $p \in K_d$ is interpreted as *restriction* of x to elements in p in its domain (resp. codomain). More precisely, $p \cdot x$ (resp. $x \cdot p$) represents the collection of x-steps which have their source(resp. target) in the subset p.

Since each of these reductions or steps are linked to elements by source and target, not *all* x-steps and y-steps are included in the element $x \cdot y$, only those for which the targets and sources match up, respectively. This is encoded in the (co-)domain axioms via $d(y) \cdot y = y$ and $x \cdot r(x) = x$. Indeed, we thereby have

$$x \cdot y = (x \cdot r(x)) \cdot (d(y) \cdot y) = x \cdot (r(x) \cdot d(y)) \cdot y,$$

meaning that the multiplication $x \cdot y$ automatically restricts x (resp. y) to the intersection $r(x) \cdot d(y)$ in its codomain (resp. domain). This can be further evidenced by the fact that multiplication is commutative in K_d, meaning that we may also write

$$x \cdot y = (x \cdot r(y)) \cdot (d(x) \cdot y).$$

In particular, if $x \cdot y = 0$, it means that none of the reductions of x may be concatenated by those of y.

The modal operators, as indicated above, tell us how collections of transitions connect different sets of objects. When the transitions are relations, the backward diamond corresponds to the diamond $\diamond_R$ defined in the previous chapter. However, when considering other models of Kleene algebra, for example those in which parallel reductions may take place, these operators "flatten" the collection of transitions x into a relational representation thereof.

The converse operation sends an element $x \in K$, *i.e.* a collection of reductions, to the collection of reductions $\overline{x}$ in the opposite direction. More precisely, for every reduction $l \to r$ in x, the reduction $r \to l$ is in $\overline{x}$.

These intuitions become concrete when we consider models of Kleene algebra in Sections 6.1.21 and 6.1.22 as relational or path algebras. The abstraction provided by Kleene algebra allow us to consider these, as well as others, at once. Abstract systems of calculation and their properties may thereby be encoded free from any concrete context.

2.2.4. Abstract rewriting in MKA. Calculatory properties such as confluence and termination can be expressed in the setting of modal Kleene algebra. These are expressed similarly to those in the relational setting, that is using inequalities in the algebra. However, since we have an algebra of transition systems, we can be more general. We first consider cases of semi-commutation before tackling confluence properties. Given $x, y \in K$, we say that the ordered pair (x, y)

i) *semi-commutes locally* if $xy \leq y^*x^*$,

ii) *semi-commutes* if $x^*y^* \leq y^*x^*$, and

iii) has the *Church-Rosser property* if $(x + y)^* \leq y^*x^*$.

Replacing x by $\overline{y}$, we have the properties of local confluence, confluence and the Church-Rosser property for the element y.

This gives further insight into consistency properties. Indeed, the semi-commutations above may be considered as *re-ordering* properties. The second property, for example, expresses that for any iteration of x-steps followed by an iteration of y-steps, we can find a reduction sequence consisting in y-steps followed by x-steps. The Church-Rosser property expresses that any reduction sequence consisting in x- and y-steps in *any order* can be re-ordered into a reduction sequence in which y-steps take place before x-steps. Confluence properties are thus special cases of re-ordering properties. These properties are diagrammatically represented as follows:

$$\begin{tikzcd}
 & \cdot \arrow[dr, "y"] & \\
\cdot \arrow[ur, "x"] \arrow[dr, "y^*"'] & & \cdot \\
 & \cdot \arrow[ur, "x^*"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \cdot \arrow[dr, "y^*"] & \\
\cdot \arrow[ur, "x^*"] \arrow[dr, "y^*"'] & & \cdot \\
 & \cdot \arrow[ur, "x^*"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[rr, leftrightarrow, "(x+y)^*"] \arrow[dr, "y^*"'] & & \cdot \\
 & \cdot \arrow[ur, "x^*"'] &
\end{tikzcd}$$

We may also express the same ideas under the modal operators. While these two notions of confluence properties coincide in the relational setting, they do not coincide in other models of Kleene algebra, such as the path model, which are of interest. Given $x, y \in K$, we say that the ordered pair (x, y)

- **i)** *modally semi-commutes locally* if $\langle x||y\rangle \leq |y^*\rangle\langle x^*|$,
- **ii)** *modally semi-commutes* if $\langle x^*||y^*\rangle \leq |y^*\rangle\langle x^*|$, and
- **iii)** has the *modal Church-Rosser property* if $|(x+y)^*\rangle \leq |y^*\rangle\langle x^*|$.

Termination is expressed in modal Kleene algebra using the forward diamond modalities. An element $x \in K$ *terminates*, or is *Noetherian*, provided that for all $p \in K_d$ the implication

$$p \leq |x\rangle p \Rightarrow p = 0,$$

holds. This is precisely the condition given for termination of relations above in Section 2.1.3 and is interpreted as such in the relational model of Kleene algebra, see Section 6.1.21.

2.2.5. Consistency. We obtain generalised versions of Newman's lemma [28] and the Church-Rosser theorem [109] in the setting of (modal) Kleene algebra. This, as in the case of ARS, provides us with a generalised consistency theorem:

6.2.11 Theorem (Consistency for MKA). *Let K be a modal Kleene algebra and $x, y \in K$ such that $(x + y) \in \mathcal{N}(K)$ and (x, y) locally modally semi-commute. Then (x, y) has the Church-Rosser property.*

CHAPTER 3.
CATEGORICAL MODEL OF CALCULATION

In the previous chapter, we considered ARS and MKA, the first a relational point of view of a single system of calculation, the second an abstract algebra of systems of calculation. Here we present another modelisation of an abstract system of calculation in the form of (higher) directed graphs. In contrast to ARS, these have labelled transitions, so we can track reduction sequences. Furthermore, parallel reductions are allowed, which means that coherence properties as well as consistency properties may be treated.

However, as we will develop later, this modelisation does not exclusively have benefits. The fact that we consider labelled transitions helps us keep track of different reduction sequences, but in turn means that we must consider each specific transition individually. A consequence of this, for example, is that we lose the point-free characterisation of consistency properties and of termination that we had in the case of ARS.

We first discuss the notion of 1-polygraph, defining calculatory properties in this context before turning to questions of coherence for these structures. We then discuss the higher dimensional case, in which higher dimensional systems of calculation are captured by the notion of n-polygraph, terminology due to Burroni [11, 12] and originally introduced as computads by Street, see [107, 108]. These structures have been used to capture higher dimensional rewriting systems presenting a diverse range of algebraic structures, and have been used, for example, to solve coherence problems in algebra [25, 46, 69], and for monoidal categories [66]. We discuss (abstract) coherence for such higher systems and show that, in this context, they are captured in terms of those given in the one-dimensional case.

3.1. GRAPHICAL TRANSITIONS

The structure we employ is that of directed psuedo-graph, which we refer to as 1-*polygraph*. Formally, a 1-polygraph P consists of a pair (P_0, P_1) of sets and two functions

$$s_0, t_0 : P_1 \to P_0,$$

called 0-*source* and 0-*target*, respectively, or simply source and target if no confusion is possible. Elements of P_0 are called 0-*cells* or *objects* while elements of P_1 are called *generating* 1-*cells*, or *reduction* or *rewrite steps*. Note that the source and target maps may not be injective, *i.e.* parallel transitions are permitted. We denote a generating 1-cell by $f : x \to y$ or

$$x \xrightarrow{f} y$$

A 1-polygraph *generates* the free 1-category P^*, also denoted by $P_0[P_1]$, whose objects, or 0-cells, are those of P and whose morphisms, or 1-cells, are formal compositions of generating 1-cells. This is related to the topological interpretation of calculation discussed in Chapter 1. Indeed, the morphisms of the free category P^* correspond to *directed* paths in the graph (P_0, P_1), and are called *reduction* or *rewriting sequences*.

We also consider the free groupoid $P^\top$, also denoted by $P_0(P_1)$, generated by P. This is the groupoid whose objects are those of P_0 and whose morphisms are formal compositions of elements of P_1 and their formal inverses. These morphisms then correspond to the *undirected* paths in (P_0, P_1), and are called *zig-zag sequences*.

3.1.1. Rewriting properties. The calculatory properties of a 1-polygraph can be explicitly defined internally to their structure, but we prefer to connect them to the relational ARS point of view. This simplifies terminology and provides clearer notions of rewriting properties, but in particular highlights that the calculatory properties of a 1-polygraph are relational; the only thing we add when considering 1-polygraphs is a well-defined spatial interpretation which allows us to keep track of reduction sequences.

To a 1-polygraph P, we associate an ARS on P_0, denoted by $\to_P$, and defined by

$$x \to_P y \qquad \Longleftrightarrow \qquad \exists u : x \to y \in P_1,$$

for all $x, y \in P_0$. The relation $\to_P$ is the "flattening" of the polygraph, in the sense that we now only have information about connectivity of 0-cells, but have lost information about distinct rewriting paths. Another way of saying this is that we have collapsed all parallel morphisms into a single abstract transition.

The rewriting properties (confluence, Church-Rosser, termination) of P are simply those of $\to_P$. In contrast to ARS, we may now talk about confluence of a single branching or zig-zag:

i) We say that a (local) branching (f, g) is *confluent* if there exists a confluence (f', g') as depicted below:

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x_1 \arrow[dr, "f'"'] & & x_2 \arrow[dl, "g'"] \\
& x' &
\end{tikzcd}$$

Such a diamond is called a *confluence diagram*.

ii) A zig-zag $h \in P_1^\top$ is *confluent* if there exists a confluence (h', k') as depicted below:

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h"] \arrow[dr, "h'"'] & & v \arrow[dl, "k'"] \\
& u' &
\end{tikzcd}$$

We say that P is *confluent* (resp. *locally confluent*) if every branching (resp. local branching) of P is confluent. We say that P is *Church-Rosser* if every zig-zag sequence of P is confluent.

This shows an important difference between the polygraphic and ARS approaches to rewriting theory. In polygraphs, rewriting properties are "point-wise" in the sense that we specify confluence of *every* branching or zig-zag when expressing consistency properties. Moreover, we are able to specify which confluence we choose to complete the branching into a confluence diagram, instead of just knowing that some confluence exists. This is a benefit, in that we have more control over reduction sequences, but makes formalisation more tricky since the objects in question must be tracked.

In the ARS setting, consistency properties are expressed via inclusions of sets. These encode the universal and existential quantifications over individual branchings and confluences, respectively, that we observe in the polygraphic setting.

3.1.2. Coherence. This point-wise nature of the polygraphic approach to abstract systems of calculation becomes primordial when describing coherence properties. Indeed, since we can manipulate the transformations themselves as if they were objects, we may treat them as such, *i.e.* consider them as being subject to "higher" reductions.

These are encoded in the notion of *cellular extension*. Given a 1-polygraph P, a cellular extension is a set Γ equipped with maps

$$\begin{tikzcd}
P_1^* & \Gamma, \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"]
\end{tikzcd}$$

where we recall that $P_1^\top$ is the set of reduction sequences in P. This means, in particular, that the pair (P_1^*, Γ) is a 1-polygraph, *i.e.* that we have a higher system of calculation whose objects are the reduction sequences, directed paths, of P, and whose reduction steps are known as 2-cells, denoted by $\alpha : f \Rightarrow g$,

$$f \xRightarrow{\alpha} g, \qquad \text{or} \qquad \begin{tikzcd}
\cdot \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D, above}] & \cdot
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

In keeping with the discussion in Chapter 1, we want the higher cells to encode equivalence between reduction paths which calculate the same equivalence between 0-cells. For this reason, we impose that the 1-source and 1-target maps satisfy *globular relations* as indicated by the right-hand diagram above, and explicitly stated in Section 5.3.2.

A cellular extension Γ consisting of a generating 2-cell $\alpha_{f,g}$ for every local branching (f, g) completed to a confluence diagram by some confluence (f', g') is called a *family of generating confluences*. This situation is diagrammatically depicted as follows:

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x_1 \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, "f'"'] & & x_2 \arrow[dl, "g'"] \\
& x' &
\end{tikzcd}$$

We say that Γ is a *homotopy basis* if for all parallel zig-zag sequences f and g in P, *i.e.* those with common source and target, there exists a 2-cell α formed from the elements of Γ connecting f and g. Symbolically and diagrammatically stated:

$$\forall f, g \in P_1^\top, \begin{cases} s_0(f) = s_0(g) \\ t_0(f) = t_0(g) \end{cases} \quad \exists \alpha \in P^\top(\Gamma)_2, \alpha : f \Rightarrow g.$$

$$\begin{tikzcd}
\cdot \arrow[rr, <->, bend left=40, "f"{name=F}] \arrow[rr, <->, bend right=40, "g"'{name=G}] & & \cdot
\arrow[from=F, to=G, phantom, "\Downarrow \alpha"]
\end{tikzcd}$$

The coherence theorem for abstract systems of calculation, stated in terms of polygraphs, affirms that every family of generating confluences constitutes a homotopy basis:

?? Theorem (Consistency for 1-polygraphs). *Let P be a locally confluent, terminating 1-polygraph. Then P has the Church-Rosser property.*

It is essentially proved in two steps, repeating the pattern we saw in the case of consistency: first a coherent version of Newman's lemma, then a coherent version of the Church-Rosser theorem.

3.2. HIGHER DIMENSIONAL REWRITING SYSTEMS

As described in Chapter 1, given an abstract system of calculation, coherence proofs by rewriting use the calculatory properties of the system to present the reduction sequences as a free system subject to certain higher relations. This process can be iterated on these new higher relations, producing yet higher relations, etc. Due to this propagation of coherence, higher dimensional rewriting systems are worth studying.

These are represented by *n-polygraphs*, also called *higher polygraphs*, introduced by Street as *computads* [106, 107] and later dubbed polygraphs by Burroni in [11], see also [13]. These have a free structure up to the last dimension, which is considered to be the "rewriting dimension", since it is the last set of higher relations added to the system. These higher polygraphs may therefore also serve as presentations of higher dimensional structures which are free up to the last dimension, see Section 5.3.

In keeping with the goal of this thesis, namely the description of coherence for abstract higher systems of calculation, we describe here that underlying such a higher structure is a 1-polygraph, and thereby an ARS, encoding all of the calculatory properties of the final dimension, *i.e.* the dimension of rewriting.

3.2.1. Higher polygraphs. In the previous section, we defined 1-polygraphs as directed pseudo-graphs. We also saw the definition of cellular extension. Combining these two notions, we can view a 1-polygraph as a cellular extension P_1 of the set P_0. This motivates the following inductive definition of higher polygraphs.

For $n \geq 0$, an *n-polygraph* P consists of a set P_0 and for every $0 \leq k < n$ a cellular extension P_{k+1} of the *free k-category*

$$P_0[P_1] \ldots [P_k].$$

At each level, we have a system of calculation P_k on the free structure $P_0[P_1] \ldots [P_{k-1}]$ on the level just below. Each of these could be considered as the level of rewriting, but implicitly, an n-polygraph is an abstract rewriting system (of dimension n) on a free system (of dimension $(n-1)$. It is thus the dimension of the final cellular extension which we consider to be the dimension of rewriting, or calculation.

3.2.2. Dimension of rewriting. However, before we associate a 1-polygraph or ARS to this dimension of rewriting, we must place the reduction rules of P_n in *context*. Indeed, due to the underlying structure being of dimension $n-1$, which we can consider to be strictly greater than zero, we must incorporate the compositions of the underlying cells of dimensions $k < n$ in order to capture the full scope of the higher dimensional rewriting system. When working with higher categories, this contextualisation of higher cells is built into the free categorical constructions.

For example, with two dimensional cells, we should not only consider the generating 2-cell on the left below, reducing a 1-cell u to a 1-cell v, but also the composite 2-cell pictured on the right reducing wuw' to wvw', for all 1-cells w, w' for which these compositions make sense.

$$\begin{tikzcd} x \arrow[r, bend left=60, "u", ""{name=U, below}] \arrow[r, bend right=60, "v"', ""{name=D}] & y \arrow[Rightarrow, from=U, to=D, "\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} x' \arrow[r, "w"] & x \arrow[r, bend left=60, "u", ""{name=U, below}] \arrow[r, bend right=60, "v"', ""{name=D}] & y \arrow[r, "w'"] & y' \arrow[Rightarrow, from=U, to=D, "f"'] \end{tikzcd}$$

We therefore consider n-cells of the form

$$f_{n-1} \star_{n-2} \ldots \star_2 (f_2 \star_1 (f_1 \star_0 \alpha \star_0 g_1) \star_1 g_2) \star_2 \ldots \star_{n-2} g_{n-1},$$

where f_i, g_i are i-cells of $P_0[P_1] \ldots [P_{n-1}]$ for $0 \leq i \leq n$, and $\alpha \in P_n$. The set of such n-cells is denoted by P_n^c. As mentioned above, this is compatible with free higher categorical constructions in the sense that any n-cell of the free n-category generated by

P can be written as an $(n-1)$-composite of cells of the above form using the laws of higher categories.

These rules in context provide us with the rewriting information at dimension n encoded by the n-polygraph P. We formalise this by defining the *underlying rewriting polygraph*, denoted P^c, whose 0-cells are the $(n-1)$-cells of the free $(n-1)$-category $P_0[P_1]\dots[P_{n-1}]$, and whose 1-cells are elements of P_n^c, *i.e.* the generating n-cells in context.

The rewriting properties of the n-polygraph are defined to be those of P^c, which in turn are defined to be those of its underlying ARS. In this way we again see that even in higher dimensional rewriting, we recover a relational system of calculation which encodes the properties at dimension n.

3.2.3. Iterated coherence. Just as consistency properties of higher dimensional rewriting systems can be seen via the underlying 1-polygraph described above, the problem of (abstract) coherence for n-polygraphs is also relegated to the problem of coherence for 1-polygraphs. Indeed, a cellular extension of the rewriting polygraph underlying an n-polygraph P can be transported "up" to dimension $(n+1)$ and be seen as a cellular extension of the free n-category generated by P, and vice versa.

Importantly, however, good rewriting properties at dimension one are propagated upwards to the higher systems of calculation we obtain when considering coherence properties. More explicitly, starting with a locally confluent and terminating 1-polygraph P, we have seen above that we can construct a set P_2, a family of generating confluences, which renders P one-dimensionally contractible. It can then be shown that the 1-polygraph (P_1^*, P_2^c) is also locally confluent and terminating, allowing us to build a coherent three-dimensional extension of the 2-polygraph (P_0, P_1, P_2). This process continues to arbitrary dimension.

This propagation of coherence via iterating the coherence theorem for one-dimensional systems of calculation allows the calculation of useful algebraic objects, cofibrant replacements, as well as providing calculatory invariants of algebraic structures [65, 67].

CHAPTER 4.
TOPOLOGICAL MODELS OF CONCURRENCY

In Chapter 1, we recalled the model of concurrent systems give by *asynchronous graphs*. While these provide a rich description of concurrency, in this chapter we will focus on topological models for concurrent systems. For more information about asynchronous graphs, we refer the reader to Section 3.3.2 of [43].

On the one hand, this is because classical models of concurrency are out of the scope of this thesis. Indeed, my work in this area is essentially limited to the study of algebraico-topological invariants for concurrent systems. On the other hand, asynchronous graphs become unwieldy when considering higher dimensional concurrent systems, especially when resources of different arities come into play [60]. Indeed, the synchronization primitives in asynchronous graphs are essentially two-dimensional.

The natural structure which responds to both of these shortcomings is that of pre-cubical sets [113]. These are combinatorial presentations of spaces built from glueing cubes of various dimensions along their borders, allowing for a higher analysis, and come equipped with a natural realisation as partially ordered spaces, thus providing a topological interpretation. We refer the interested reader to [117] and [58] for a description of classical models for concurrency, their relationships, and relationships with pre-cubical set-based models for concurrency defined below.

In Section 4.1.1, we recall the notion of cubical complexes and describe their geometric realisation. We then quickly move on to the more general setting of directed and partially ordered spaces in Section 4.2.

4.1. CUBICAL COMPLEXES

4.1.1. Pre-cubical sets. A *pre-cubical set* K is a family $(K_n)_{n\in\mathbb{N}}$ of sets along with maps $\partial_i^\alpha : K_n \to K_{n-1}$ with $1 \leq i \leq n$ and $\alpha \in \{0,1\}$. These are called *face maps*, and must satisfy the following *cubical relations*:

$$\partial_i^\alpha \circ \partial_j^\beta = \partial_{j-1}^\beta \circ \partial_i^\alpha.$$

This is a purely combinatorial description of a space built from cubes of various dimensions. The elements of K_n are thought of as n-dimensional cubes, and the face maps encode how lower-dimensional cubes constitute faces of higher ones. The cubical relations encode the "cubical shape" of these abstract elements. In order to have a spatial interpretation of these structures, we will define a geometric realisation using a standard n-cube:

Let $\square_n = \{(t_1, ..., t_n) \in \mathbb{R}^n \mid \forall 1 \leq i \leq n, 0 \leq t_i \leq 1\}$ the standard n-cube in $\mathbb{R}^n$.
We define for $\alpha \in \{0,1\}$, $n \in \mathbb{N}$, $1 \leq i \leq n+1$, $\rho_i^\alpha : \square_n \to \square_{n+1}$ by:

$$\rho_i^\alpha(t_1, ..., t_n) = (t_1, ..., t_{i-1}, \alpha, t_i, ..., t_n)$$

4.1.2. Geometric realisation. Let K be a pre-cubical set. Equipping each set K_n with the discrete topology, we form the topological space $R(K) = \coprod_{n\in\mathbb{N}} K_n \times \square_n$, where we take the usual induced topology on standard n-cubes, the product topology on $K_n \times \square_n$ and the disjoint topology over the disjoint union. The elements of $R(K)$ are then pairs $(e, \vec{a})$, e being a standard n-dimensional cube in K and $\vec{a} \in [0,1]^n$.

The *geometric realisation* of K, denoted by $Geom(K)$, is the quotient space obtained from $R(K)$ under the least equivalence relation $\equiv$ such that:

$$\forall \alpha \in \{0,1\},\ n \in \mathbb{N},\ 1 \leq i \leq n,\ x \in K_n,\ t \in \square_{n-1},\ (\partial_i^\alpha(x), t) \equiv (x, \rho_i^\alpha(t)).$$

The advantage of using pre-cubical sets is that they are more expressive when it comes to modelling resources with arities larger than one, *i.e.* resources which are not mutexes.

4.2. DIRECTED AND PARTIALLY ORDERED SPACES

Directed topology has been originally introduced as a model, and a tool, for studying and classifying concurrent systems, in computer science [57, 95]. In this approach, the possible states of several processes running concurrently are modeled as points of a topological space of configurations, in which executions are described by paths. Thus restricted areas appear when these processes have to synchronise, to perform a joint task, or to use a shared object that cannot be shared by more than a certain number of processes.

It is natural to study the homotopical and homological properties of this configuration space in order to deduce some interesting properties of the parallel programs involved, for verification purposes, or for classifying synchronisation primitives. A usual model for concurrent processes is actually the one of higher-dimensional automata, that are based on (pre-)cubical sets, and are the most expressive known models in concurrency theory [113].

Contrarily to ordinary algebraic topology, the invariants of interest are invariants under some form of continuous deformation, but which has to respect the flow of time. In short, the only valid homotopies are the ones which never invert the flow of time. For mathematical developments and some applications we refer to the two books [43, 61]. Other topological models for concurrency exist, in particular *streams* [80] and local po-spaces [44], which generalise pospaces to looping situations. In this thesis we will focus exclusively on the interpretation offered by directed spaces.

4.2.1. Directed spaces. Recall from [61] that a *dispace*, or a *dispace* for short, is a pair $\mathcal{X} = (X, dX)$, where X is a topological space and dX is a set of *paths* in X, i.e., continuous maps from $[0, 1]$ to X, called *directed paths*, of *dipaths* for short, such that every constant path is directed, and dX is closed under monotonic reparametrization and concatenation.

4.2.2. Partially ordered spaces. Partially-ordered spaces, or pospaces for short, form particular dispaces : these are topological spaces X equipped with a partial order $\leq$ on X which is closed under the product topology. The directed structure is thus given by continuous maps $p\ : [0, 1] \to X$ such that $p(s) \leq p(t)$, for all $s \leq t$ in $[0, 1]$.

Another useful class of dispaces is given by the directed geometric realization of finite pre-cubical sets defined in the previous section. These are made of gluings of cubical cells, on which the dispace structure is locally that of a particular partially-ordered space : each n-dimensional cell is identified with $[0, 1]^n$ ordered component-wise. This last class is in particular very useful in applications to concurrency and distributed systems theory, see e.g. [43].

4.2.3. Classification of directed spaces. Classifying directed spaces according as semantic models of concurrent processes requires a fine analysis. As an example, we have depicted two dispaces in Figure 4.1, which are built as the gluing of squares, the white ones, each of which is equipped with the product order on $\mathbb{R}^2$. They are the directed geometric realisation of certain pre-cubical sets, *i.e.* are higher-dimensional automata in the sense of [95]. They are not dihomeomorphic spaces since they are already non homotopy equivalent: the fundamental group of the leftmost one, which we call X, is the free abelian group on three generators, whereas the fundamental group of the rightmost one, Y, is the free abelian group on four generators.

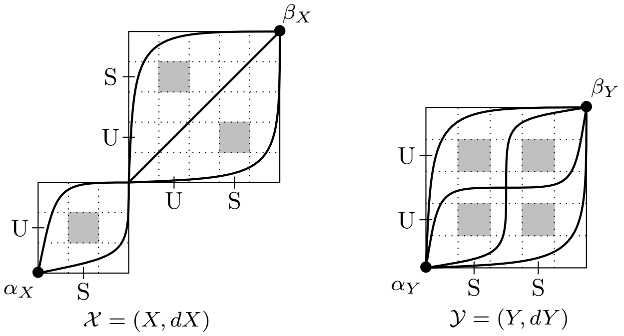


Figure 4.1: Examples of pospaces coming from non-equivalent concurrent programs.

4.2.4. Spaces of paths. We would like not only to distinguish the underlying spaces, but the dipaths in each. Indeed, it is these which link these spaces to concurrency, classifying the various execution properties of the considered concurrent system. For this, we look at spaces of directed paths between points in the underlying space.

Consider now the topological space of directed paths, with the compact-open topology, from the lowest point of X (resp. Y), denoted by α_X (resp. α_Y), to the highest point of X (resp. Y), denoted by β_X (resp. α_Y). The topological space $\overrightarrow{Di}(\mathcal{X})(\alpha_X, \beta_X)$ of directed paths from α_X to β_X, is homotopy equivalent to a six point space, corresponding to the six dihomotopy classes of dipaths pictured in Figure 4.1.

However, these two dispaces should not be considered as equivalent in the sense that they correspond to distinct concurrent programs. Therefore comparing spaces of dipaths exclusively between two particular points in each space is not sufficient for distinguishing these dispaces.

4.2.5. Natural homotopy and natural homology. Directed topological invariants, most notably the computationally tractable ones such as homology, have been long in the making, starting again with [57]. In this thesis, we focus on *natural homotopy* and *natural homotopy*, see [31]. The intuition behind these invariants is to encode the way in which the homotopy or homology types of the spaces of directed paths vary when we move the end points.

Indeed, with the possibility of considering all the directed path spaces, we can distinguish the two former pospaces. Indeed, if we consider the space of directed paths between α and β, as in Figure 4.2, it has the homotopy type of a discrete space with four points but we can show that, in Y, there is no pair of points between which we have a directed path space with the same homotopy type.

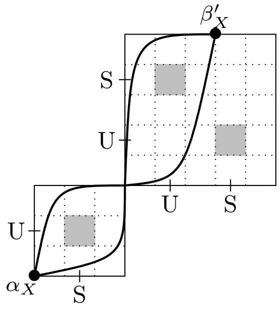


Figure 4.2: Changing the base points to exhibit a particular space of directed paths.

The algebraic structures which log all of the homotopy or homology types of the directed path spaces between each pair of points is that of a natural system, see Section 11.2.1, hence the appellation of these invariants. These first appeared in as generalised coefficients in the homology theories of categories, see also . Importantly, natural homotopy and homology not only keep track of the homotopy or homology types of each space of directed paths, respectively, but also the group morphisms obtained from inclusions of these spaces given by extensions, as described in Section 1.5.9.

PART I.
FORMALISING COHERENCE

CHAPTER 5.
DIMENSIONS OF REWRITING

In this preliminary section, we recall rewriting paradigms formally. First, in Section 5.1, we recall the definition of, constructions with, and calculatory properties as expressed in a relational setting, which we call *abstract rewriting*. A more detailed account of abstract rewriting can be found in the standard references [4, 111]. We then do the same in the context of 1-polygraphs in Section 5.2, offering a spatial interpretation of abstract rewriting.

Next, we distinguish abstract rewriting from string rewriting and recall the description of the latter via 2-polygraphs in Section 5.3, before turning to the higher dimensional case in Section 5.4. In these, we also recall standard definitions and properties of higher categories. The combinatorial structures we employ to describe higher dimensional rewriting systems, known as *polygraphs* [11] or *computads* [107], are more fully treated in [13] and [68] from the point of view of calculation, and their relation to the folk model structure on higher categories may be found in [91].

This chapter contains no new results and serves to provide context for later chapters.

5.1. ABSTRACT REWRITING

Before addressing the polygraphic approach to systems of calculation, we recall their description and properties as binary relations over a set. This is on the one hand to introduce terminology, and on the other it will provide intuition for the description of ARS via Kleene algebra.

5.1.1. Abstract rewriting systems. We consider a set X, whose elements we refer to as *objects*. An *abstract rewriting system* (ARS) R is a *(binary) relation* on X, *i.e.* a subset $R \subseteq X \times X$. We interpret

$$(x, y) \in R$$

as the statement: x is *rewritten* to y. In the spirit of rewriting theory, we will henceforth denote a relation R by $\to_R$ or simply $\to$ when no confusion is possible. In this notation, $(x, y) \in R$ will be replaced by $x \to y$.

5.1.2. The algebra of relations. The set of (binary) relations over X is denoted by $Rel(X)$. The *composition of relations* is denoted by $\circ$ and is defined, for $\rightarrow_R, \rightarrow_S \in Rel(X)$, by

$$\rightarrow_R \circ \rightarrow_S := \{(x,z) \mid \exists y \in X,\ x \rightarrow_R y \text{ and } y \rightarrow_S z\}.$$

The neutral element of this operation is the *diagonal of* X, *i.e.* the set

$$\{(x,x) \mid\ x \in X\} \in Rel(X),$$

which will henceforth be denoted by Δ_X, or simply Δ when no confusion is possible. The *converse* of $\rightarrow \in Rel(X)$ is denoted by $\leftarrow$. A relation $T \subseteq \Delta$ is called a *subidentity*. Subsets of X are in bijective correspondence with subidentities by the map which sends a subset $P \subseteq X$ to the subidentity $P_\Delta := \{(x,x) \mid x \in P\}$.

We additionally equip $Rel(X)$ with the operation of *set-union*, denoted by $\cup$, whose neutral element is the empty set, denoted as usual by $\emptyset$. The structure $(Rel(X),\ \cup,\ \emptyset\ \circ,\ \Delta)$, called the *total relation algebra over* X, forms a semiring, see Section 6.1. The subidentities constitute a subalgebra of $Rel(X)$ and is denoted by $SubId(X)$. In this subalgebra, composition of relations corresponds to intersection, making $SubId(X)$ isomorphic to the power-set lattice of X. Explicitly, for subsets P, P' of X,

$$(x,x) \in P_\Delta \circ P'_\Delta \qquad \Longleftrightarrow \qquad x \in P \cap P'.$$

Relations on X act on the algebra of subidentities via modal operators. For a relation R, we define the *forward diamond operator* of R by

$$\begin{aligned} |R\rangle : SubId(X) &\longrightarrow SubId(X) \\ P_\Delta &\longmapsto \{(x,x) \mid \exists y \in X,\ (x,y) \in R \wedge y \in P\}. \end{aligned}$$

This operator sends a subset P of X to the points accessible from P by R-steps. It is a modal operator in the sense of . Generalisations thereof will be important when defining rewriting properties in the setting of Kleene algebra.

5.1.3. Iteration and equivalence. Let $\rightarrow$ be an ARS over X. In order to express rewriting properties, we consider the following relations generated by $\rightarrow$:

- The i^{th} *iteration of* $\rightarrow$, inductively defined by

$$\rightarrow^0 := \Delta \qquad\qquad \rightarrow^i := \rightarrow^{i-1} \circ \rightarrow, \quad \forall i > 0,$$

- the *transitive closure of* $\rightarrow$, defined by

$$\overset{+}{\rightarrow} := \bigcup_{i>0} \rightarrow^i,$$

- the *reflexive closure of* $\rightarrow$, defined by

$$\leftrightarrow := \leftarrow \cup \rightarrow,$$

- the *reflexive, transitive closure of* $\to$, defined by

$$\overset{*}{\to} := \bigcup_{i>0} \to^{i},$$

- the *reflexive, transitive closure of* $\to$, defined by

$$\overset{*}{\leftrightarrow} := (\leftarrow \cup \to)^{*} = \bigcup_{i \geq 0} (\leftarrow \cup \to)^{i},$$

  This is also called the *zig-zag relation* generated by $\to$.

These are called closures because they coincide with the smallest relation on X containing $\to$ and having the indicated property (refliexivity, transitivity, ...). When $x \overset{*}{\to} y$, we say that there is a *rewriting* or *reduction sequence* or *path* from x to y. When $x \overset{*}{\leftrightarrow} y$ we say that x and y are *equivalent under* $\to$, or that there is a *zig-zag sequence* or *path* from x to y.

A *normal form* of an ARS $\to$ is an element $x \in X$, such that for all $y \in X$, $(x, y) \not\in \to$. In other words, it is a maximal element for the relation $\to$. Given $x \in X$, a *normal form of* x is an element y such that there exists a rewriting path from x to y, *i.e.* $x \overset{*}{\to} y$, and y is a normal form. When no confusion is possible, we will denote a normal form of an object x by $\hat{x}$.

The zig-zag relation generated by an ARS $\to$ on a set X is an equivalence relation. It describes the partitioning the set X into the classes of objects which are equal under the system of calculation represented by the ARS. We denote the quotient of X by the equivalence relation $\overset{*}{\leftrightarrow}$ by $\overline{X}$ or $X/\overset{*}{\leftrightarrow}$.

5.1.4. Abstract rewriting properties. Now we define rewriting properties for ARS. First, we introduce notation and terminology for relations generated by an ARS $\to$ and its inverse $\leftarrow$. These describe certain configurations, or shapes, of interest when studying properties of the ARS represented by $\to$.

- The *local branching relation*, denoted by $\swarrow\searrow$, is defined by

$$x \swarrow\searrow y \qquad \Longleftrightarrow \qquad (x, y) \in \leftarrow \circ \to .$$

- the *(global) branching relation*, denoted by $\swarrow\!*\!\searrow$, is defined by

$$x \swarrow\!*\!\searrow y \qquad \Longleftrightarrow \qquad (x, y) \in \overset{*}{\leftarrow} \circ \overset{*}{\to} .$$

- the *confluence relation*, denoted by $\searrow^{*}\swarrow$, is defined by

$$x \searrow^{*}\swarrow y \qquad \Longleftrightarrow \qquad (x, y) \in \overset{*}{\to} \circ \overset{*}{\leftarrow} .$$

The (local) branching relation represents a (local) choice in the method of calculation. Indeed, when x and y are related by the (local) branching relation, they are rewritten from a common element. Conversely, the confluence relation represents a reconciliation in the system of calculation represented by the ARS in the sense that when x and y are related by the confluence relation, they are rewritten to the same element.

i) An ARS $\rightarrow$ is *confluent* (resp. *locally confluent*) if the branching relation (resp. local branching relation) is included in the confluence relation, *i.e.*

$$\swarrow{}_{*}\searrow \subseteq \searrow^{*}\swarrow \qquad\qquad (\text{resp.}\ \swarrow\searrow \subseteq \searrow^{*}\swarrow).$$

ii) We say that an ARS is *Church-Rosser* or has the *Church-Rosser property* if the zig-zag relation is included in the confluence relation, *i.e.*

$$\overset{*}{\leftrightarrow} \subseteq \searrow^{*}\swarrow.$$

iii) An ARS $\rightarrow$ *terminates*, is *terminating* or is *Noetherian* if there are no infinite rewrite paths. For relations, this property can be described by the following predicate:

$$\forall P \subseteq X, P_{\Delta} \subseteq |R\rangle(P_{\Delta}) \qquad \Rightarrow \qquad P = \emptyset.$$

iv) An ARS is *convergent* when it is both confluent and terminating.

5.1.5. Interpretation as properties of calculation. Confluence is a central notion in abstract rewriting theory. Indeed, the inclusion means that the ARS is consistent in the sense that whenever elements x and y are rewritten from a common element, they will eventually be rewritten to the same element. In particular, the system of calculation represented by the ARS is deterministic: starting at some element of X and calculating *in any way* will either produce a unique answer, or an infinite rewrite path. In other words, this means that normal forms are unique.

5.1.6. Lemma ([4]). *Let* $\rightarrow \subseteq X \times X$ *be a confluent ARS and* $x, y, y' \in X$. *If* y, y' *are normal forms for* x, *then* $y = y'$.

The inclusion required to fulfil the Church-Rosser property states another important property of calculation, namely that whenever elements x and y are equivalent under the rules of calculation, there exist (*directed*) rewrite paths, *i.e.* calculations, describing this equivalence by reducing x and y to a common element in their equivalence class.

It expresses a seemingly stronger consistency property than confluence, in that it requires that not only conflicting calculations will eventually be resolved, but that equivalence itself is captured by calculation. However, one of the most classical theorems in abstract rewriting theory, the Church-Rosser theorem, Theorem 5.1.10, states that this property is in fact equivalent to that of confluence.

The third property of calculatory import is that of termination. Indeed, the predicate above is interpreted as the statement: if P has no R-maximal element, P must be empty. This assures the non-existence of infinite reduction sequences. In particular, it also guarantees the existence of normal forms.

5.1.7. Lemma ([4]). *Let $\to \subseteq X \times X$ be a terminating ARS and $x \in X$. Then x admits at least one normal form.*

Convergence, *i.e.* confluence and termination, therefore represents the ideal situation for an ARS representing a system of calculation. Indeed, it follows from the above that for a convergent ARS, we have both existence and uniqueness of normal forms. This means that the system of calculation represented by the ARS is will compute a unique answer from any starting point.

5.1.8. Consistency theorems. In the following chapters, we will discuss coherence properties for ARS. As described in the introduction, these rely on higher dimensional cells encoding equivalences between rewrite paths. Here, we discuss Newman's lemma and the Church-Rosser theorem as proto-coherence theorems, resulting in what we call the *consistency theorem* in this thesis.

As discussed in the previous paragraph, convergence of an ARS is a strong consistency property. As we will see in following sections, the schema of coherence proofs is the following: from local coherence and termination one proves global coherence in terms of branchings, and then in terms of zig-zags. We describe this schema here in the context of consistency for ARS.

First, we abuse terminology in this paragraph in order to mirror the procedure of coherence proofs. A terminating ARS is said to be

i) *locally, directedly consistent* when it is locally confluent,

ii) *directedly consistent* when it is confluent,

iii) and *consistent* when it has the Church-Rosser property.

Newman's lemma takes local directed consistency to global directed, consistency:

5.1.9. Theorem (Newman's lemma [92]). *Let $\to$ be an ARS. If $\to$ is locally confluent and terminates, then $\to$ is confluent.*

From global directed consistency, we obtain global consistency via the Church-Rosser theorem:

5.1.10. Theorem (Church-Rosser theorem [22]). *An ARS $\to$ is confluent, if, and only if, it is Church-Rosser.*

For the above, more modern proofs may be found in e.g. [4].

Thus, the *consistency theorem* for ARS states that a locally confluent and terminating ARS has the Church-Rosser property.

5.1.11. Theorem (Consistency for ARS). *Let $\to$ be a locally confluent, terminating ARS. Then $\to$ is Church-Rosser.*

It is an immediate consequence of the preceding theorems, and we highlight it only to compare it to coherence theorems. Indeed, this procedure, taking local consistency properties to global consistency properties under the hypothesis of termination, will be echoed in subsequent chapters when we treat coherence properties of ARS.

This theorem thus describes how local information can be propagated to a 0-dimensional calculatory coherence property, *i.e.* consistency, in the sense that an ARS with the Church-Rosser property computes equivalences. Indeed, given equivalent elements, this property tells us that we can reduce them to the same normal form, which is unique by confluence and exists by termination.

Furthermore, it shows that we can use the ARS to decide equivalence. Indeed, as a consequence of the previous result, for a normalising and locally confluent ARS, to decide equivalence of two elements x and y, it suffices to check the syntactical equality of their normal forms $\hat{x}$ and $\hat{y}$. If the normal forms are computable and the syntactic identity is decidable then the equivalence is decidable.

5.1.12. Remark. One of the main benefits of the relational description of calculation is that all of the properties are defined as simple inclusions and take place internally to an algebra with a simple signature. Furthermore, the inclusion defining, for example confluence, allows us to complete *all* branchings by confluences all at once. This contrasts the polygraphic case, in which we must quantify universally over individual branchings and existentially over individual confluences. This point-free description of calculatory properties makes it an ideal setting for handling consistency properties, but is balanced by the disadvantage of losing information about rewrite paths.

Indeed, when composing an ARS with itself, we lose information about the intermediary points. For example, if $(x, y), (x, y') \in R$ and $(y, z), (y', z) \in S$, the distinct reduction sequences $xRySz$ and $xRy'Sz$ are identified. This can be summed up by saying that $(x, z) \in R \circ S$ implies only that x is rewritten to z via *some* path consisting in an R-step followed by an S-step. In this sense, relations provide information about *connectedness* or *equivalence* of objects, but forget the *choices* made while rewriting.

In the following final paragraphs of this first section, we explore alternative relational models for ARS which attempt to address these issues.

5.1.13. Multirewriting systems.

Given sets X and Y, recall from that a *multirelation* M on $X \times Y$ is a map

$$M : X \times Y \longrightarrow \overline{\mathbb{N}},$$

where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. A *multirewriting system* over a set X is a multirelation on $X \times X$.

Multirewriting systems over X are equipped with algebraic operations via those of $\overline{\mathbb{N}}$. For multirewriting systems m_1, m_2 over X,

- **i)** Addition: $(m_1 + m_2)(x, y) = m_1(x, y) + m_2(x, y)$. The neutral element for addition is the multirelation $\emptyset_{multi}$ which sends every pair (x, y) to 0.
- **ii)** Composition: $(m_1; m_2)(x, y) = \Sigma_{y \in X} m_1(x, y) \times m_2(y, z)$. The neutral element for composition is the multirelation m_Δ which sends every pair (x, y) to 1.

The set of multirewriting systems over X, denoted by $mRel(X)$, forms a semiring under these operations. Furthermore, we have a homomorphism

$$\begin{aligned} \lfloor - \rfloor : mRel(X) &\longrightarrow Rel(X) \\ m &\longmapsto \lfloor m \rfloor, \end{aligned}$$

where $\lfloor m \rfloor$ is defined by $(x, y) \in \lfloor m \rfloor$ if, and only if, $m(x, y) > 0$. The rewriting properties (confluence, termination,…) of a multirelation m are defined to be those of the associated relation $\lfloor m \rfloor$.

While multirelations allow us to consider several transitions between objects, these are not labelled precisely enough to conserve information about how rewriting paths are built from the initial multirewriting system.

5.1.14. Polyrewriting systems. We define a *polyrewriting system* on a set X to be a family

$$\{\rightarrow_i\}_{i \in I}$$

of binary relations on X. For brevity, a polyrewriting system will be denoted by $\rightrightarrows_I$, or simply by $\rightrightarrows$ when no confusion is possible. We denote the set of polyrewriting systems on X by $pRel(X)$.

The advantage of considering such objects is that they allow us to consider several *parallel* rewriting steps, *i.e.* those which relate the same pair of elements but should be considered as distinct: for $i, j \in I$ with $i \neq j$, we may have $x \rightarrow_i y$ and $x \rightarrow_j y$. In contrast to multirewriting systems, these are labelled by the corresponding index.

A natural way of equipping polyrewriting systems with a notion of composition, is simply to lift the composition of relations to the level of sets:

$$\rightrightarrows_I \circ \rightrightarrows_J := \{\rightarrow_i \circ \rightarrow_j\}_{(i,j) \in I \times J}.$$

Note however, that there is not a unique identity element for this operation. Indeed, any polyrewriting system $\{\rightarrow_i\}_{i \in I}$ such that $\cup_{i \in I} \rightarrow_i = \Delta$ is an identity.

For the addition operation, we have two choices:

$$\rightrightarrows_I + \rightrightarrows_J := \{\rightarrow_k\}_{k \in I \coprod J} \quad \text{or} \quad \rightrightarrows_I \cup \rightrightarrows_J := \{\rightarrow_i \cup \rightarrow_j\}_{(i,j) \in I \times J}.$$

The first consists in taking the union of the families $\rightrightarrows_I$ and $\rightrightarrows_J$, and the second is the lifting of the addition operation in $Rel(X)$ to the set level.

However, we observe that there is a natural map sending polyrewriting systems to multirewriting systems on some set X:

$$\begin{aligned} m_{(-)} : pRel(X) &\longrightarrow mRel(X) \\ \rightrightarrows_I &\longmapsto m_I, \end{aligned}$$

where m_I sends the pair (x, y) to the natural number $m_I(x, y) = |\{i \in I \mid x \rightarrow_i y\}|$.

This map commutes with the composition operations. Denoting by $m_{I\times J}$ the image of $\rightrightarrows_I \circ \rightrightarrows_J$, we have

$$\begin{aligned} m_{I\times J}(x,z) &= |\{(i,j) \in I \times J \mid x \to_i \circ \to_j z\}| \\ &= |\{(i,j) \in I \times J \mid \exists y \in X, x \to_i y \wedge y \to_j z\}| \\ &= \Sigma_{y\in X} |\{(i \in I \mid x \to_i y\}| \times |\{(j \in J \mid y \to_j z| \\ &= (m_I; m_J)(x,z). \end{aligned}$$

Similarly, this map commutes with the addition operation +. Furthermore, any element which is an identity for the composition of polyrewriting systems is sent to the identity element m_Δ for multirewriting systems.

The rewriting properties of a polyrewriting system (confluence, termination, . . .) are defined to be those of its underlying multirewriting system.

5.1.15. Remark. As discussed in Remark 5.1.12, relational algebra give us no way of expressing the existence of parallel transitions, and furthermore, no way of accessing distinct rewriting paths. While poly- and multirelations are indeed enrichments of the notion of (binary) relations, they fall short of providing a satisfactory model of calculatory systems.

Indeed, in the context of multirewriting, we can encode the fact that there are several parallel transitions between two given elements. However, similarly to the case of binary relations, we have no way of distinguishing between parallel paths, nor of accessing the rewriting paths. This is essentially due to the fact that transitions are not labelled. So while multirelations equip us with a notion of parallelism, they have the same problem as classical relations with respect to labels

The other relational approach described above, polyrewriting, attempts to solve this issue by labelling binary relations with an indexing set. This allows us to distinguish between different interleavings of compositions of the individual relations, for example $\to_i \circ \to_j$ and $\to_j \circ \to_i$. However, since this is essentially a lifting of the structure of binary relations to the power-set level, we still lose information about distinct rewriting paths, as exemplified in Remark 5.1.12. Furthermore, the algebraic properties of polyrelations is not clear, nor easy to track.

In order to study coherence properties, some amount of access to specific rewriting paths. We therefore turn to directed graphs, in which not only the vertices, but also the edges, are labelled. This allows us to distinguish parallel rewriting paths as distinct words in the labels. However, since we keep track of this information, we lose the point-free, relatively simple expression of rewriting properties given in relational algebras.

5.2. ABSTRACT REWRITING WITH 1-POLYGRAPHS

Polygraphs are combinatorial objects which generate free (higher) categories. The term polygraph comes from Street's *computads* [106, 107], later dubbed *polygraphs* by

Burroni [11, 12]. Here, we will discuss the description of abstract rewriting systems as 1-polygraphs. First, we recall the necessary definitions of categories.

5.2.1. Categories. To fix notation, we recall that a *(small, strict)* 1-*category* $\mathcal{C}$, or simply *category* when no confusion is possible, consists of the following data:

i) Sets $\mathcal{C}_0$ and $\mathcal{C}_1$ along with maps as in the following diagram:

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\iota_1" description] & \mathcal{C}_1 \arrow[l, shift right=3, "s_0"'] \arrow[l, shift left=3, "t_0"]
\end{tikzcd}.$$

Elements of $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) are called 0-*cells* or *objects* (resp. 1-*cells* or *morphisms*). The map ι_1 is called the *unit* map, and for $c \in \mathcal{C}_0$, we denote by 1_c the image of c under ι_1. This is called the *identity* on c. The maps s_0 and t_0 are known as 0-*source* and 0-*target*, or simply as source and target when no confusion is possible. These satisfy the *identity relations*:

$$s_0 \circ \iota_1 = id_{\mathcal{C}_0} \qquad \text{and} \qquad t_0 \circ \iota_1 = id_{\mathcal{C}_0}.$$

ii) A partial *composition* operation, denoted by $\star_0$. For 1-cells f and g, their composite, denoted by $f \star_0 g$, is defined provided that $t(f) = s(g)$. This may be seen as an operation

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \longrightarrow \mathcal{C}_1,$$

where $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 = \{(f,g) \mid f, g \in \mathcal{C}_1, t(f) = s(g)\}$. We have the following *identity laws* for composition:

$$f \star_0 1_{t(f)} = f \qquad \text{and} \qquad 1_{s(f)} \star_0 f = f,$$

and source and target maps interact with composition by

$$s(f \star_0 g) = s(f) \qquad \text{and} \qquad t(f \star_0 g) = t(g).$$

The set of 1-cells with source x and target y will be denoted by $\mathcal{C}(x,y)$ and is called a *hom-set*. We will often denote the composition $f \star_0 g$ by fg when no confusion is possible. Finally, we say that 1-cells f and f' are *parallel* when $s(f) = s(f')$ and $t(f) = t(f')$, *i.e.* when f and f' belong to the same hom-set.

A 1-cell f of a category $\mathcal{C}$ is *invertible* if there exists a 1-cell g such that

$$f \star_0 g = 1_{s(f)} \qquad \text{and} \qquad g \star_0 f = 1_{t(f)}.$$

The 1-cell g is then unique, and is called the *inverse* of f. It is denoted by f^{-1} or f^{-}. A category in which every 1-cell is invertible is called a *groupoid*.

5.2.2. 1-polygraphs. A 1-polygraph P consists of a pair (P_0, P_1) of sets and two functions

$$s_0, t_0 : P_1 \to P_0,$$

called *source* and *target*, respectively. Such a structure is also referred to as a *quiver* or *directed pseudo-graph.* Elements of P_0 are called 0-*cells* or *objects* while elements of P_1 are called *generating* 1-*cells*, or *reduction* or *rewrite steps.*

5.2.3. Free constructions. A 1-polygraph *generates* the free 1-category P^*, also denoted by $P_0[P_1]$, whose objects, or 0-cells, are those of P and whose morphisms, or 1-cells, are formal compositions of generating 1-cells.

More precisely, we form the free finite words on the set $P_1 \cup \{1_x\}_{x \in P_0}$, where the 1_x are formal identity cells. We then eliminate words that don't correspond to well-formed compositions, *i.e.* we consider the set $\tilde{P}_1$ consisting of words

$$f_1 f_2 \cdots f_k,$$

on $P_1 \cup \{1_x\}_{x \in P_0}$ such that $t_0(f_i) = s_0(f_{i+1})$ for all $1 \leq i < k$. We then take the quotient of $\tilde{P}_1$ by the equivalence relation $\simeq$ generated by the relation containing

$$f_1 \cdots f_i 1_{t(f_i)=s(f_{i+1})} f_{i+1} \cdots f_k \to f_1 \cdots f_i f_{i+1} \cdots f_k. \tag{5.2.4}$$

for all permitted sequences of generating 1-cells $f_1, \dots, f_k$. The relation (5.2.4) is the *congruence* on $\tilde{P}_1$ generated by the relation containing $f 1_{t_0(f)} \to f$ and $1_{s_0(f)} f \to f$ for all generating 1-cells.

The quotient set $\tilde{P}/\simeq$ is denoted by P_1^*. The pair (P_0, P_1^*) is naturally equipped with the structure of a category.

i) We extend source and target maps freely on $\tilde{P}_1$ by

$$s_0(f_1 f_2 \cdots f_k) = s_0(f_1) \quad \text{and} \quad t_0(f_1 f_2 \cdots f_k) = t_0(f_k).$$

These maps are compatible with the equivalence relation $\simeq$, so we obtain induced maps $s_0, t_0 : P_1^* \to P_0$. The unit map takes a 0-cell x to the 1-cell 1_x.

ii) The concatenation of words in $\tilde{P}_0$, subject to the condition that source and targets coincide correctly, passes to the quotient $\simeq$. We denote the resulting operation by $\star_0$.

The morphisms of this category are called *reduction* or *rewriting sequences* or *paths* in the 1-polygraph P.

We also consider the free groupoid generated by $P = (P_0, P_1)$. This is the groupoid whose objects are those of P_0 and whose morphisms are formal compositions of elements of P_1 and their formal inverses, quotiented by another equivalence relation $\cong$. The set of such permitted sequences of morphisms and formal inverses is denoted here by $\hat{P}_1$.

Given a generating 1-cell f, we denote its formal inverse by f^-. The equivalence relation $\cong$ is the symmetric, reflexive, transitive closure of the relation containing (5.2.4) and the pairs

$$f_1 \cdots f_i f f^- f_{i+1} \cdots f_k \to f_1 \cdots f_i f_{i+1} \cdots f_k \tag{5.2.1}$$
$$f_1 \cdots f_i f^- f f_{i+1} \cdots f_k \to f_1 \cdots f_i f_{i+1} \cdots f_k \tag{5.2.2}$$

for all permitted sequences of 1-cells. This corresponds to the congruence generated on $\hat{P}_1$ by the relation containing the pairs $f 1_{t_0(f)} \to f$, $1_{s_0(f)} f \to f$, $f f^- \to 1_{s_0(f)}$ and $f^- f \to 1_{t_0(f)}$ for all generating 1-cells f.

The quotient set $\hat{P}_1/\cong$ is denoted by $P_1^\top$, and we define the structure of a groupoid on the pair $(P_0, P_1^\top)$ similarly to the case of the free category construction. This groupoid is denoted by $P^\top$ or $P_0(P_1)$. The morphisms of this groupoid are called *zig-zag sequences* or *paths*.

In the case of 1-polygraphs, we will denote a 1-cell $f \in P^*$ (resp. $f \in P^\top$) with source x and target y by $f : x \to y$ (resp. $f : x \leftrightarrow y$) or by

$$x \xrightarrow{f} y \qquad \text{(resp. } x \xleftrightarrow{f} y \text{).}$$

We denote by $\overline{P}_0$ or P_0/P_1 the quotient of the set P_0 by the equivalence relation $\equiv$ given by connected components in P. Formally, $x \equiv y$ if, and only if, there exists a zig-zag sequence from x to y. This is the *equivalence* underlying the system of calculation described by the 1-polygraph in question.

5.2.5. Underlying (poly)relation. A relation $\to$ defines a 1-polygraph $(X, \to)$ in which source and target are given by the projections onto each coordinate. This gives a 1-polygraph in which transitions are labelled by pairs (x, y) and in which there are no parallel paths: between any two 0-cells there is at most one transition. However, when considering coherence properties, see Section 7, we will require manipulating parallel reductions, *i.e.* those with the same source and target. Furthermore, to a 1-polygraph we associate an underlying ARS. Polygraphs thus provide a finer description of calculatory systems than ARS.

Given a 1-polygraph $P = (P_0, P_1)$, we define an underlying polyrewriting system $\rightrightarrows_P$ on P_0 given by

$$\rightrightarrows_P = \{\to_f\}_{f\ \in P_1}, \qquad \text{where} \quad \to_f = \{(s_0(f), t_0(f))\}.$$

That is, we take the family of singleton relations given by generating 1-cells of P. Note that since there are no external algebraic operations on polygraphs, *i.e.* composition, addition of polygraphs, we cannot talk about this map being a homomorphism.

We then also obtain an associated multirewriting system m_P, and its underlying relation $\lfloor m_P \rfloor$. The latter, which will henceforth be denoted by $\to_P$, is given by

$$x \to_P y \qquad \Longleftrightarrow \qquad \exists u : x \to y \in P_1,$$

for all $x, y \in P_0$. The relation $\to_P$ is the “flattening” of the polygraph, in the sense that we now only have information about connectivity of 0-cells, but have lost information about distinct rewriting paths. Another way of saying this is that we have collapsed all parallel morphisms into a single abstract transition.

5.2.6. Rewriting properties of 1-polygraphs. The rewriting properties of a 1-polygraph P are those of its underlying polyrewriting system $\rightrightarrows_P$, and therefore those of the ARS $\to_P$. In this way, we obtain notions of (local) confluence, termination and consistency in the context of 1-polygraphs, see Section 5.1.

We obtain 1-polygraphic versions of Theorems 5.1.9 and 5.1.10 by applying the same to the underlying ARS. We also obtain an analog of Theorem 5.1.11 for 1-polygraphs as an immediate consequence. These results are expressed in the more general setting of higher dimensional rewriting systems in Corollaries 5.4.14, 5.4.15 and 5.4.16, respectively.

5.3. String rewriting

A *string rewriting system* (SRS) is an abstract rewriting system on a free monoid [8] or a free 1-category [68]. While abstract rewriting sequences describe directed subsystems of an equivalence relation, string rewriting systems describe a directed subsystem of *congruences*, *i.e.* equivalence relations which respect the underlying algebraic structure. SRS can be described in the structure of a 2-polygraph, considering the 1-cells as elements of the monoid or category, and 2-cells as the system of calculation. We recall the notion of 2-polygraph here and relate SRS to abstract consistency, showing that the underlying structure can be used to transport certain shapes into algebraic contexts. This leads to the critical branching lemma, refining the consistency check for SRS.

5.3.1. 2-categories as enriched categories. A 2-*category* $\mathcal{C}$ consists of a set $\mathcal{C}_0$ of 0-*cells*, and for all $x, y \in \mathcal{C}_0$, a 1-category $\mathcal{C}(x, y)$, called the *hom-category* associated to the pair (x, y). The 0- and 1-cells of hom-categories are called the 1- and 2-cells of $\mathcal{C}$, respectively. The composition in $\mathcal{C}(x, y)$ is called 1-*composition*.

These sets are equipped with algebraic laws providing the underlying structure of a 1-category.

i) For all 0-cells x, y, z, we have a functor

$$c_{x,y,z} : \mathcal{C}(x, y) \times \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, z),$$

called the 0-*composition*.

ii) For every 0-cell x we have a specified 0-cell of $\mathcal{C}(x, x)$, denoted by 1_x and called the *identity* 1-*cell* on x.

This data is required to satisfy associativity laws: for all $x, y, z, t \in \mathcal{C}_0$,

$$c_{x,z,t} \circ (c_{x,y,z} \times Id_{\mathcal{C}(z,t)}) = c_{x,y,t} \circ (Id_{\mathcal{C}(x,y)} \times c_{y,z,t}),$$

and unit laws: for all $x, y \in \mathcal{C}_0$,

$$c_{x,x,y} \circ (1_x \times Id_{\mathcal{C}(x,y)}) = c_{x,y,y} \circ (Id_{\mathcal{C}(x,y)} \times 1_y)$$

where 1_x and 1_y denote the constant functors equal to 1_x and 1_y respectively.

In the language of (higher) category theory, this is equivalent to saying that a 2-category is a category enriched in categories. Indeed, a category $\mathcal{C}$ is *enriched* in a category $\mathcal{V}$ when every hom-set $\mathcal{C}(x, y)$ of $\mathcal{C}$ is an object of $\mathcal{V}$.

5.3.2. 2-categories as globular sets. In this thesis, we prefer to view 2-categories as globular 2-sets with additional structure. This optic is equivalent to the definition as enriched categories, but provides valuable diagrammatic intuitions, which is why we develop it explicitly here.

A 2-category $\mathcal{C}$ consists of the following data:

i) A reflexive 2-globular set, that is, a diagram of sets and functions of the form

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\iota_1" description] & \mathcal{C}_1 \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "t_0"] \arrow[r, "\iota_2" description] & \mathcal{C}_1 \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "t_1"]
\end{tikzcd}$$

Elements of $\mathcal{C}_i$ are called *i-cells.* The maps $s_i, t_i : \mathcal{C}_{i+1} \to \mathcal{C}_i$, called *i-source* and *i-target* respectively, satisfy the *globular relations*:

$$s_i \circ s_{i+1} = s_i \circ t_{i+1}, \qquad t_i \circ s_{i+1} = t_i \circ t_{i+1} \tag{5.3.3}$$

The globular relations (5.3.3) imply that any 2-cell f has a globular shape:

$$\begin{tikzcd}
s_0 \circ s_1(f) = s_0 \circ t_1(f) \arrow[r, bend left, "s_1(f)"] \arrow[r, bend right, "t_1(f)"'] \arrow[r, phantom, "f \Downarrow"] & t_0 \circ s_1(f) = t_0 \circ t_1(f)
\end{tikzcd}$$

ii) The map ι_i, the *i-unit map*, interacts with sources and targets via the *unit laws*:

$$s_i \circ \iota_{i+1} = id_{\mathcal{C}_i}, \quad t_i \circ \iota_{i+1} = id_{\mathcal{C}_i}. \tag{5.3.4}$$

As before, the identity 1-cell (resp. 2-cell) on a 0-cell x (resp. 1-cell u) is denoted by 1_x (resp. 1_u). Note that every 0-cell x has a unique identity 2-cell $1_{1_x} = \iota_2 \circ \iota_1(x)$. Graphically, these identities are represented, respectively, by

$$\begin{tikzcd}
x \arrow[loop above, "1_x"]
\end{tikzcd} \qquad \begin{tikzcd}
x \arrow[r, "u"'] \arrow[r, phantom, bend left=60, "\overset{1_u}{\circlearrowright}"] & y
\end{tikzcd} \qquad \begin{tikzcd}
x \arrow[loop above, Rightarrow, "1_{1_x}"]
\end{tikzcd}$$

iii) The pairs $(\mathcal{C}_0, \mathcal{C}_1)$ and $(\mathcal{C}_1, \mathcal{C}_2)$ have the structure of 1-categories; we denote their composition operations by $\star_0$ and $\star_1$, respectively.

As a consequence of globularity, the pair $(\mathcal{C}_0, \mathcal{C}_2)$ is naturally equipped with the structure of a 1-category. Indeed, we define source and target maps as the composites $s_0 \circ s_1$ and $t_0 \circ t_1$. These are still denoted s_0 and t_0, respectively, abusing notation. The 0-composition of 2-cells, denoted by $\star_0$, extends that of $(\mathcal{C}_0, \mathcal{C}_1)$ via globularity, in the sense described by the following diagram:

$$\begin{tikzcd}
p \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow f"] & q \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow g"] & r
\end{tikzcd}$$

We have the usual categorical laws for interaction between 0-source and 0-target in $(\mathcal{C}_0, \mathcal{C}_2)$, namely

$$s_0(f \star_0 g) = s_0(f) \qquad \text{and} \qquad t_0(f \star_0 g) = t_0(g),$$

but, as the diagram indicates, the 1-source and 1-target are homomorphisms for the 0-composition of 2-cells in the sense that

$$s_1(f \star_0 g) = s_1(f) \star_0 s_1(g) \qquad \text{and} \qquad t_1(f \star_0 g) = t_1(f) \star_0 t_1(g).$$

iv) The composition operations $\star_0$ and $\star_1$ satisfy the *strict interchange law*

$$(f \star_1 f') \star_0 (g \star_1 g') = (f \star_0 g) \star_1 (f' \star_0 g'), \tag{5.3.5}$$

for all $0 \leq j < k < n$, and whenever all compositions are defined. In the diagrammatic notation, the interchange law (5.3.5), is represented by

$$\begin{tikzcd}
 & & & & & & x \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow f"] & y \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow g"] & z \\
x \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Downarrow f"] \arrow[r, phantom, shift right=2ex, "\Downarrow f'"] & y & \star_0 & y \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Downarrow g"] \arrow[r, phantom, shift right=2ex, "\Downarrow g'"] & z & = & & \star_1 & \\
 & & & & & & x \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow f'"] & y \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow g'"] & z
\end{tikzcd}$$

Note that whenever the expression on the left of (5.3.5) is defined, the right side is too. However, the converse does not hold, due to compositions of the form

$$\begin{tikzcd}
x \arrow[r] \arrow[rr, bend left=50, "\Downarrow f"'] \arrow[r, bend right=70, "\Downarrow f'"] & y \arrow[r] \arrow[rr, bend right=50, "\Downarrow g'"] & z \arrow[r] \arrow[r, bend left=70, "\Downarrow g"'] & w,
\end{tikzcd}$$

for which $(f \star_0 g) \star_1 (f' \star_0 g')$ is a well-formed 2-cell, whereas the 1-compositions $f \star_1 f'$ and $g \star_1 g'$ are not defined.

For more details on 2-categories, both in the enriched description and that using globular sets, we invite the reader to consult [89, XII. 3.]. In the rest of this thesis, we will take this point of view on 2-categories, although we remind the reader that the two definitions are equivalent.

5.3.6. Notation. We abuse notation, denoting by $\mathcal{C}_1$ the underlying 1-category $(\mathcal{C}_0, \mathcal{C}_1)$ of $\mathcal{C}$. The $(k-1)$-composition of k-cells f and g is denoted by juxtaposition fg, and the $(k-1)$-source $s_{k-1}(f)$ and the $(k-1)$-target $t_{k-1}(f)$ of a k-cell f are denoted by $s(f)$ and $t(f)$, respectively. To highlight the relative dimensions of cells, we denote 1-cells by single arrows $\to$ and 2-cells by double arrows $\Rightarrow$.

5.3.7. Whiskers. In order to define the composition of 1- and 2-cells, we use the identity 2-cells on 1-cells. Let u, v be 1-cells and f a 2-cell such that $t_0(u) = s_0(f)$ and $t_0(f) = s_0(v)$. The *whiskering* of f by u (on the left) and v (on the right) is the composite $1_u \star_0 f \star_0 1_v$. In diagrammatic notation, this is represented by

$$\begin{tikzcd}
s_0(u) \arrow[r, "u", "\circlearrowright\, 1_u"'] & t_0(u)=s_0(f) \arrow[r, bend left=50, "s_1(f)", ""{name=U, below}] \arrow[r, bend right=50, "t_1(f)"', ""{name=D}] & t_0(f)=s_0(v) \arrow[r, "v", "\circlearrowright\, 1_v"'] & t_0(v) \arrow[Rightarrow, from=U, to=D, "f"']
\end{tikzcd}$$

To simplify notation, we denote this k-cell by ufv, and will diagrammatically represent such a cell as above but without the identity 2-cells 1_u and 1_v.

5.3.8. $(2,1)$-categories and 2-groupoids. In a 2-category, a k-cell f of $\mathcal{C}$ is i-*invertible* if there exists a k-cell g in $\mathcal{C}$ with i-source $t_i(f)$ and i-target $s_i(f)$ in $\mathcal{C}$ called the *i-inverse of f*, which satisfies

$$f \star_i g \;=\; 1_{s_i(f)} \qquad \text{and} \qquad g \star_i f \;=\; 1_{t_i(f)}.$$

The i-inverse of a k-cell is necessarily unique. When $i = k-1$, we say that $f : u \to v$ is *invertible* and we denote its $(k-1)$-inverse by $f^{-1} : v \to u$ or $f^{-} : v \to u$, which we simply call its *inverse.*

A $(2,1)$-*category* is a 2 category in which all 2-cells are invertible. In the language of enriched categories, this is equivalent to saying that it is a category enriched in groupoids. Similarly, a 2-groupoid, also called a $(2,0)$-category, is a 2-category in which *all* cells are invertible, *i.e.* a groupoid enriched in groupoids.

5.3.9. Cellular extensions. To better define 2-polygraphs, we first define cellular extensions of categories, and the resulting free constructions and quotients. Cellular extensions are a generalisation of the notion of relations in the setting of categories. We fix a small category $\mathcal{C}$.

A *sphere* in $\mathcal{C}$ is a pair (f, g) of parallel 1-cells. For a sphere (f, g), we say that f is its *source* and that g is its *target.* A category is *aspherical* if all of its spheres are of the form (f, f). A *cellular extension* of $\mathcal{C}$ is a subset of the spheres in $\mathcal{C}$. This is formally encoded by a set Γ equipped with a map ∂ to the set of spheres in $\mathcal{C}$.

$$\mathcal{C}_1 \xleftarrow{\;\partial\;} \Gamma$$

For $\alpha \in \Gamma$, the *boundary* of the sphere $\partial(\alpha)$ is denoted $(s_1(\alpha), t_1(\alpha))$, so we may think of a cellular extension as a set of generating 2-cells $\alpha : f \Rightarrow g$, where f and g are parallel 1-cells of $\mathcal{C}$. We naturally obtain two maps $s_1, t_1 : \Gamma \to \mathcal{C}_1$ satisfying the globular relations

$$s_0 \circ s_1 = s_0 \circ t_1 \qquad \text{and} \qquad t_0 \circ s_1 = t_0 \circ t_1.$$

Analogously, we can define cellular extensions of 2-categories as subsets of 2-spheres, *i.e.* parallel 2-cells, see Section 5.4.8.

5.3.10. Free constructions and quotients. Given a cellular extension Γ of $\mathcal{C}$, we construct the *free 2-category* $\mathcal{C}[\Gamma]$ *generated by* Γ *over* $\mathcal{C}$, which has the same 0- and 1-cells as $\mathcal{C}$, and in which 2-cells are formal composites of elements of Γ. In other words, each hom-category $\mathcal{C}[\Gamma](x, y)$, where x, y are 0-cells of $\mathcal{C}$, is the free category generated from the 1-polygraph with 0-cells $\mathcal{C}(x, y)$ and generating 1-cells $\partial^{-1}(\mathcal{C}(x, y))$.

The *quotient* $\mathcal{C}/\Gamma$ of $\mathcal{C}$ by a cellular extension Γ is the category in which we identify 1-cells which are the source and target of some element of Γ. This quotient has the structure of a 1-category because of the globular shape of the elements of Γ.

We similarly define the quotient 2-category of some 2-category by a cellular extension. Given a 1-cell f or a 2-cell α, we denote their equivalence classes in a quotient category by $\overline{f}$ and $\overline{\alpha}$ respectively.

The notion of quotient leads to another free construction resulting in a $(2, 1)$-category. Given a cellular extension Γ of $\mathcal{C}$, we define Γ^- to be the cellular extension in which the source and target of spheres in Γ have been inverted. Formally, for each element $\alpha : f \Rightarrow g$ of Γ, we have a sphere $\alpha^- : g \Rightarrow f$ in Γ^-. We then construct the free 2-category $\mathcal{C}[\Gamma, \Gamma^-]$, where 2-cells are formal composites of spheres in Γ and their formal inverses. Then, we define a cellular extension $Inv(\Gamma)$ of $\mathcal{C}[\Gamma, \Gamma^-]$ consisting of 3-cells

$$\alpha \star_1 \alpha^- \Rrightarrow 1_f \text{ and } \alpha^- \star_1 \alpha \Rrightarrow 1_g$$

for every $\alpha : f \Rightarrow g$ of Γ. The quotient of $\mathcal{C}[\Gamma, \Gamma^-]$ by this extension, $\mathcal{C}(\Gamma)$, is the *free* $(2, 1)$*-category generated by* Γ *over* $\mathcal{C}$:

$$\mathcal{C}(\Gamma) := \mathcal{C}[\Gamma, \Gamma^-]/Inv(\Gamma).$$

Equivalently, each hom-category $\mathcal{C}(x, y)$ is the free groupoid generated by the 1-polygraph with 0-cells $\mathcal{C}(x, y)$ and generating 1-cells $\partial^{-1}(\mathcal{C}(x, y))$.

5.3.11. 2-Polygraphs. We now have the vocabulary to introduce 2-polygraphs as presentations of categories. We have already introduced 1-polygraphs as a generalisation of directed graphs, but they fit into an inductive definition of higher structures as follows: a 0-polygraph is a set P_0, and a 1-polygraph is a directed (pseudo-)graph (P_0, P_1). The latter may be seen as a cellular extension of the set P_0. This observation leads to the definition of 2-polygraph.

A 2-*polygraph* is a triple $P = (P_0, P_1, P_2)$ where P_2 is a cellular extension of P^*. This corresponds to the following diagram in the category of sets:

$$\begin{tikzcd}
P_0 & P_1^* \arrow[l, shift right, "t_0"'] \arrow[l, shift left, "s_0"'] & \\
 & P_1 \arrow[u] \arrow[ul, shift left, "\overline{t}_0"] \arrow[ul, shift right, "\overline{s}_0"'] & P_2 \arrow[ul, shift right, "t_1"'] \arrow[ul, shift left, "s_1"]
\end{tikzcd}$$

Here we have denoted the source and target maps of P by $\overline{s}_0, \overline{t}_0$ in order to notationally distinguish them from their free extensions s_0, t_0 to P^*. In general, however, we do not notationally distinguish the source and target maps associated to a cellular extension from those obtained in the free constructions. The maps s_1, t_1 are called the 1-*source* and 1-*target*, respectively, and elements of P_2 are called *generating* 2-*cells*.

A 2-polygraph $P = (P_0, P_1, P_2)$ generates free 2-categorical structures. Firstly, the *free* 2-*category generated over* P, denoted by P^* and defined by

$$P^* := P_1^*[P_2].$$

We also define the *free* $(2,1)$-*category over* P. Denoted $P^\top$, this is the free $(2,1)$-category generated by P_2 over P_1^*, *i.e.*

$$P^\top := P_1^*(P_2) = P_1^*[P_2, P_2^-]/Inv(P_2).$$

5.3.12. Presentations of categories. We fix a 2-polygraph $P = (P_0, P_1, P_2)$.

The *category presented by* P, given by

$$\overline{P} := P^*/P_2.$$

We say that P *presents* a category $\mathcal{C}$ when $\overline{P} \cong \mathcal{C}$. In this case, the set of 0-cells of $\mathcal{C}$ is P_0, and we say that P_1 is the set of *generating* 1-*cells* of $\mathcal{C}$, while the elements of P_2 are called its *relations*. For a 2-cell $\alpha : f \Rightarrow g$ of P, we consider its class in $\overline{P}$ to be $\overline{\alpha} = \overline{f} = \overline{g}$.

5.3.13. Rewriting with 2-polygraphs. We can view a polygraphic presentation of a category $\mathcal{C}$ as a rewriting system on the 1-cells of $\mathcal{C}$. However, we must place the generating 2-cells in context to generate a congruence relation. This is implicit in the free category construction, as will be made explicit below. We fix a 2-polygraph P.

The underlying *rewriting* 1-*polygraph* associated to P, denoted by P^c, is given by (P_1^*, P_2^c), where

$$P_2^c = \{u\alpha v | u, v \in P_1^*, \alpha \in P_2, t_0(u) = s_0(\alpha) \wedge t_0(\alpha) = s_0(v)\}.$$

The set P_2^c is the congruence on P_1^* generated by elements of P_2, also called the *cells of* P_2 *in context*. The source and target maps are those of P^*, *i.e.* $s(u\alpha v) = us_1(\alpha)v$ and $t(u\alpha v) = ut_1(\alpha)v$. Due to laws of categorical algebra, most notably the strict exchange law, any 2-cell in the free category generated by P can be uniquely written as a 1-composite of 2-cells in context, that is of the form $u\alpha v$, see [65, Prop. 2.1.5].

The rewriting properties (confluence, termination, ...) of a 2-polygraph are those of its underlying rewriting 1-polygraphWe recover an ARS $\Rightarrow_P$ from a 2-polygraph P. Indeed, $\Rightarrow_P$ is the underlying ARS of the rewriting 1-polygraph P^c associated to P.

5.3.14. Branchings in SRS. In the setting of 2-polygraphs, a *branching* of P is a pair (α, β) of elements of P_2^* with common source. This may be represented by either of the following diagrams:

$$\begin{tikzcd}
x \arrow[r, "u" description] \arrow[r, bend left=60, "v"] \arrow[r, bend right=60, "w"'] \arrow[r, phantom, shift left=4ex, "\Uparrow\alpha"] \arrow[r, phantom, shift right=4ex, "\Downarrow\beta"] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
& u \arrow[dl, Rightarrow, bend right, "\alpha"'] \arrow[dr, Rightarrow, bend left, "\beta"] & \\
v & & w
\end{tikzcd}$$

As before, the *source* of a branching (α, β) is their common source u. We say that a branching (α, β) is *local* when α and β are rewriting steps. A local branching is *aspherical* when it is of the form (α, α), and is *Peiffer* when it is of the form

$$\begin{tikzcd}
x \arrow[r, "u"'] \arrow[r, bend left=70, "u'"] \arrow[r, phantom, shift left=3ex, "\Uparrow\alpha"] & y \arrow[r, "v"] \arrow[r, bend right=70, "v'"'] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv \arrow[dl, Rightarrow, bend right, "\alpha v"'] \arrow[dr, Rightarrow, bend left, "u\beta"] & \\
u'v & & uv'
\end{tikzcd}$$

with $\alpha : u \Rightarrow u'$ and $\beta : v \Rightarrow v'$. The remaining local branchings are called *overlapping branchings*. Overlapping branchings, as their name indicates, reduce two different 1-cells which overlap non-trivially, as indicated in the following diagram:

$$\begin{tikzcd}
x \arrow[r, "l"'] \arrow[rr, bend left=50, ""] \arrow[rr, phantom, shift left=4ex, "\Uparrow\alpha"] & y \arrow[r, "m" description] \arrow[rr, bend right=50, ""'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\beta"] & z \arrow[r, "r"] & w
\end{tikzcd}$$

The above diagram shows an overlapping branching $(\alpha r, l\beta)$ on the 1-cell rml. We equip these overlapping branchings with an order $(\alpha, \beta) \prec (u\alpha v, u\beta v)$ for all 1-cells u, v for which the above makes sense. The overlapping branchings which are minimal with respect to this order are called the *critical branchings*.

5.3.15. The critical branching lemma. The only rewriting property of a 2-polygraph P not described by its underlying ARS $\Rightarrow_P$ is the following notion of confluence relative to critical branchings. We say that P is *critically confluent* when its critical branchings are confluent.

Following the general schema of rewriting theory, that is using local information to deduce global properties, thereby reducing the number of checks required to assure consistency of the entire system, the notion of critical branching is exploited in string rewriting theory to further reduce the checks involved. Indeed, critical confluence is sufficient to deduce local confluence.

5.3.16. Theorem (Critical branching lemma [68]). *Let P be a 2-polygraph. Then P is critically confluent if, and only if, it is locally confluent.*

We also obtain analogs of Theorems 5.1.9 and 5.1.10 for 2-polygraphs by considering the ARS underlying their underlying rewriting 1-polygraphs, see Corollaries 5.4.14 and 5.4.15. In particular, we obtain a finer consistency theorem for 2-polygraphs, reducing the consistency check to critical, rather than local, branchings.

5.3.17. Theorem (Critical consistency). *Let P be a critically confluent, terminating 2-polygraph. Then P is Church-Rosser.*

5.4. HIGHER-DIMENSIONAL REWRITING

Here we recall notions of higher-dimensional abstract rewriting. In particular, we recall the definition of n-polygraphs [12], also called *computads* in [106], see also [67, 91], and their properties as rewriting systems presenting higher-dimensional categories. As described in Section 5.4.12, the final dimension of such structures is implicitly the dimension of abstract rewriting. We start with definitions and terminology for higher categories and refer to standard textbooks for details [86, 89].

5.4.1. Higher categories. Let n be a natural number. A *(strict globular) n-category* $\mathcal{C}$ consists of the following data.

i) A reflexive n-globular set, that is, a diagram of sets and functions of the form

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\iota_1" description] & \mathcal{C}_1 \arrow[l, shift right=3, "s_0"'] \arrow[l, shift left=3, "t_0"] \arrow[r, "\iota_2" description] & \cdots \arrow[l, shift right=3, "s_1"'] \arrow[l, shift left=3, "t_1"] \arrow[r, "\iota_{n-1}" description] & \mathcal{C}_{n-1} \arrow[l, shift right=3, "s_{n-2}"'] \arrow[l, shift left=3, "t_{n-2}"] \arrow[r, "\iota_n" description] & \mathcal{C}_n \arrow[l, shift right=3, "s_{n-1}"'] \arrow[l, shift left=3, "t_{n-1}"]
\end{tikzcd}$$

whose functions $s_i, t_i : \mathcal{C}_{i+1} \to \mathcal{C}_i$ and $\iota_i : \mathcal{C}_{i-1} \to \mathcal{C}_i$ satisfy the *globular relations*

$$s_i \circ s_{i+1} = s_i \circ t_{i+1}, \qquad t_i \circ s_{i+1} = t_i \circ t_{i+1} \tag{5.4.2}$$

and the *identity relations*

$$s_i \circ \iota_{i+1} = id_{\mathcal{C}_i}, \quad t_i \circ \iota_{i+1} = id_{\mathcal{C}_i}. \tag{5.4.3}$$

ii) It is equipped with a structure of category on

$$\begin{tikzcd}
\mathcal{C}_k & \mathcal{C}_\ell \arrow[l, shift right=2, "s_k^\ell"'] \arrow[l, shift left=2, "t_k^\ell"]
\end{tikzcd}$$

for all $k < \ell$, where

$$s_k^\ell := t_k \circ \ldots \circ t_{\ell-2} \circ t_{\ell-1} \qquad \text{and} \qquad t_k^\ell := s_k \circ \ldots \circ s_{\ell-2} \circ s_{\ell-1},$$

and whose *k-composition morphism* on $\mathcal{C}_\ell$ is denoted by $\star_k^\ell : \mathcal{C}_\ell \times_{\mathcal{C}_k} \mathcal{C}_\ell \to \mathcal{C}_\ell$, where $\mathcal{C}_\ell \times_{\mathcal{C}_k} \mathcal{C}_\ell$ is the set of paris of k-composable ℓ-cells, *i.e.* pairs $(f, g) \in \mathcal{C}_\ell \times \mathcal{C}_\ell$ such that $t_k^\ell(f) = s_k^\ell(g)$.

iii) The 2-globular set

$$\begin{tikzcd} \mathcal{C}_j & \mathcal{C}_k \arrow[l, shift right, "s_j^k"'] \arrow[l, shift left, "t_j^k"] & \mathcal{C}_\ell \arrow[l, shift right, "s_k^\ell"'] \arrow[l, shift left, "t_k^\ell"] \end{tikzcd}$$

is a 2-category for all $j < k < \ell$, see Section 5.3.2.

5.4.4. Terminology and notation. The elements of $\mathcal{C}_k$ are called *k-cells of $\mathcal{C}$*. For $0 \leq k < n$, we abuse notation, denoting by $\mathcal{C}_k$ the underlying k-category of k-cells of $\mathcal{C}$. We further abuse notation by denoting maps s_k^ℓ and t_k^ℓ by s_k and t_k for all $0 \leq \ell < n$, and similarly $\star_k^\ell$ will simply be denoted by $\star_k$. The maps s_i, t_i and ι_i are called *i-source*, *i-target* and *i-unit* maps, respectively. For a k-cell f of $\mathcal{C}$ and for $0 \leq i < k$, we call $s_i(f)$ (resp. $t_i(f)$) the *i-source* (resp. *i-target*) of f. We denote the identity $(k+1)$-cell of $\iota_{k+1}(f)$ by 1_f. When f and g are *i-composable* k-cells, for $i < k$, that is when $t_i(f) = s_i(g)$, we denote their i-composite by $f \star_i g$. By condition **iii)**, the composition operations satisfy the *interchange law*

$$(f \star_j f') \star_i (g \star_j g') = (f \star_i g) \star_j (f' \star_i g'), \tag{5.4.5}$$

for all $0 \leq i < j < n$, and whenever all compositions are defined.

The $(k-1)$-composition of k-cells f and g is denoted by juxtaposition fg, and the $(k-1)$-source $s_{k-1}(f)$ and the $(k-1)$-target $t_{k-1}(f)$ of a k-cell f are denoted by $s(f)$ and $t(f)$, respectively. To highlight the relative dimensions of cells, we denote them by single arrows $\to$, double arrows $\Rightarrow$, and triple arrows $\Rrightarrow$. In particular, if we denote a k-cell in $\mathcal{C}$ by $f : u \Rightarrow v$, then we denote $(k-1)$-cells of $\mathcal{C}$ by $u : p \to q$ the and the $(k+1)$-cells of $\mathcal{C}$ by $A : f \Rrightarrow g$ in to distinguish their dimensions notationally. Such globular cells are depicted as follows:

$$\begin{tikzcd}[column sep=huge] p \arrow[r, bend left=60, "u", ""{name=U, below}] \arrow[r, bend right=60, "v"', ""{name=V, above}] & q \arrow[Rightarrow, from=U, to=V, shift right=4ex, "f"', ""{name=F, right}] \arrow[Rightarrow, from=U, to=V, shift left=4ex, "g", ""{name=G, left}] \arrow[Rrightarrow, from=F, to=G, "A"] \end{tikzcd}$$

The globular relations (5.4.2) imply that any k-cell f has a globular shape with respect to its i- and j-sources and targets for $0 \leq i < j < n$:

$$\begin{tikzcd}[column sep=huge]
s_i\circ s_j(f)=s_i\circ t_j(f) \arrow[r, bend left=50, "s_j(f)"{name=U}] \arrow[r, bend right=50, "t_j(f)"'{name=D}] & t_i\circ s_j(f)=t_i\circ t_j(f) \arrow[Rightarrow, from=U, to=D, "f"']
\end{tikzcd}$$

In diagrams, the interchange law (5.4.5) is illustrated by:

$$\begin{tikzcd}
p \arrow[r, bend left=60, ""{name=A}] \arrow[r, ""{name=B}] \arrow[r, bend right=60, ""{name=C}] & q \arrow[Rightarrow, from=A, to=B, "f"] \arrow[Rightarrow, from=B, to=C, "f'"]
\end{tikzcd}
\star_i
\begin{tikzcd}
q \arrow[r, bend left=60, ""{name=A}] \arrow[r, ""{name=B}] \arrow[r, bend right=60, ""{name=C}] & r \arrow[Rightarrow, from=A, to=B, "g"] \arrow[Rightarrow, from=B, to=C, "g'"]
\end{tikzcd}
=
\begin{array}{c}
\begin{tikzcd}
p \arrow[r, bend left=50, ""{name=A}] \arrow[r, bend right=50, ""{name=B}] & q \arrow[r, bend left=50, ""{name=C}] \arrow[r, bend right=50, ""{name=D}] & r \arrow[Rightarrow, from=A, to=B, "f"] \arrow[Rightarrow, from=C, to=D, "g"]
\end{tikzcd} \\
\star_j \\
\begin{tikzcd}
p \arrow[r, bend left=50, ""{name=A}] \arrow[r, bend right=50, ""{name=B}] & q \arrow[r, bend left=50, ""{name=C}] \arrow[r, bend right=50, ""{name=D}] & r \arrow[Rightarrow, from=A, to=B, "f'"] \arrow[Rightarrow, from=C, to=D, "g'"]
\end{tikzcd}
\end{array}$$

5.4.6. Identities and whiskers. Given a k-cell f, the identity l-cell on f for $k \leq l \leq n$ is denoted by $\iota_k^l(f)$ and defined by induction, setting $\iota_k^k(f) := f$ and $\iota_k^l(f) := 1_{\iota_k^{l-1}}$ for $k < l \leq n$. In this way, for $0 \leq k < l \leq n$, we associate a unique identity cell $\iota_k^l(f)$ of dimension l to every k-cell f, which is called the *l-dimensional identity* on f.

In higher category theory, the use of such iterated identities is necessary for defining compositions between cells of different dimension. For $0 \leq i < k < l \leq n$, a k-cell f and a l-cell g such that $t_i(f) = s_i(g)$, the i-composite of f and g is defined as

$$f \star_i g = \iota_k^l(f) \star_i g.$$

If $t_i(g) = s_i(f)$, we define $g \star_i f = g \star_i \iota_k^l(f)$.

For $0 \leq i < j < k$, an (i, j)-*whiskering* of a k-cell f is a k-cell $\iota_j^k(u) \star_i f \star_i \iota_j^k(v)$, where u and v are j-cells, as in the following diagram:

$$\begin{tikzcd}[column sep=large]
s_i(u) \arrow[r, "u"] \arrow[r, phantom, shift right=12, "\underset{\iota_j^k(u)}{\circlearrowleft}"] & s_i(f) \arrow[r, bend left=50, "s_j(f)"{name=U}] \arrow[r, bend right=50, "t_j(f)"'{name=D}] & t_i(f) \arrow[Rightarrow, from=U, to=D, "f"'] \arrow[r, "v"] \arrow[r, phantom, shift right=12, "\underset{\iota_j^k(v)}{\circlearrowleft}"] & t_i(v)
\end{tikzcd}$$

To simplify notation, we denote this k-cell by $u \star_i f \star_i v$. A $(k-1, k-1)$-whiskering $1_u \star_{k-1} f \star_{k-1} 1_v$ of a k-cell f is called a *whiskering* of f and denoted by ufv.

5.4.7. (n, p)-categories. If $\mathcal{C}$ is an n-category and $0 \leq i < k \leq n$, a k-cell f of $\mathcal{C}$ is *i-invertible* if there exists a k-cell g in $\mathcal{C}$ with i-source $t_i(f)$ and i-target $s_i(f)$ in $\mathcal{C}$ called the *i-inverse of f*, which satisfies

$$f \star_i g \;=\; 1_{s_i(f)} \qquad \text{and} \qquad g \star_i f \;=\; 1_{t_i(f)}.$$

The i-inverse of a k-cell is necessarily unique. When $i = k-1$, we say that $f : u \to v$ is *invertible* and we denote its $(k-1)-$inverse by $f^{-1} : v \to u$ or $f^{-} : v \to u$ for short, which we simply call its *inverse*. If in addition the $(k-1)$-cells u and v are invertible, then there exist k-cells

$$u^{-} \star_{k-2} f^{-} \star_{k-2} v^{-} : u^{-} \to v^{-}, \qquad v^{-} \star_{k-2} f^{-} \star_{k-2} v^{-} : u^{-} \to v^{-}$$

in $\mathcal{C}$. For a natural number $p \leq n$, or for $p = n = \infty$, an (n,p)*-category* is an n-category whose k-cells are invertible for every $k > p$. In the language of enriched categories, when $n < \infty$, this is a p-category enriched in $(n-p)$-groupoids and, when $n = \infty$, is a p-category enriched in ∞-groupoids.

5.4.8. Spheres and cellular extensions. Let $\mathcal{C}$ be an n-category. A 0-*sphere* of $\mathcal{C}$ is a pair of 0-cells of $\mathcal{C}$. For $1 \leq k \leq n$, a k*-sphere* of $\mathcal{C}$ is a pair (f,g) of k-cells such that $s_{k-1}(f) = s_{k-1}(g)$ and $t_{k-1}(f) = t_{k-1}(g)$. We denote by $\mathrm{Sph}_k(\mathcal{C})$ the set of k-spheres of $\mathcal{C}$.

A *cellular extension* of $\mathcal{C}$ is a set Γ equipped with a map $\partial : \Gamma \to \mathrm{Sph}_n(\mathcal{C})$. For $\alpha \in \Gamma$, the *boundary* of the sphere $\partial(\alpha)$ is denoted $(s_n(\alpha), t_n(\alpha))$, defining in this way two maps $s_n, t_n : \Gamma \to \mathcal{C}_n$ satisfying the following globular relations

$$s_{n-1} \circ s_n = s_{n-1} \circ t_n \qquad \text{and} \qquad t_{n-1} \circ s_n = t_{n-1} \circ t_n.$$

The free $(n+1)$*-category over* $\mathcal{C}$ *generated by the cellular extension* Γ is the $(n+1)$-category, denoted by $\mathcal{C}[\Gamma]$ and defined as follows:

- **i)** its underlying n-category is $\mathcal{C}$,
- **ii)** its $(n+1)$-cells are built as formal i-compositions, for $0 \leq i \leq n$, of elements of Γ and k-cells of $\mathcal{C}$, seen as $(n+1)$-cells with source and target in $\mathcal{C}_n$.

The second point is formalised by each hom-category $\mathcal{C}_n(f,g)$ being generated freely by the 1-polygraph with 0-cells the elements of $\mathcal{C}_n(f,g)$ and whose generating 1-cells are $\partial^{-1}(\mathcal{C}_n(f,g))$, see Section 5.2.3.

The *quotient* of the n-category $\mathcal{C}$ by Γ, denoted by $\mathcal{C}/\Gamma$, is the n-category we obtain from $\mathcal{C}$ by identifying the n-cells $s_n(\alpha)$ and $t_n(\alpha)$, for every n-sphere α of Γ.

The free $(n+1,n)$*-category over* $\mathcal{C}$ *generated by* Γ, denoted by $\mathcal{C}(\Gamma)$, is defined by

$$\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \Gamma^{-}]/\mathrm{Inv}(\Gamma)$$

where

- **i)** Γ^{-} is the cellular extension of $\mathcal{C}$ made of spheres $\alpha^{-} = (t_n(\alpha), s_n(\alpha))$, for each α in Γ,

- **ii)** $\mathrm{Inv}(\Gamma)$ is the cellular extension of the free $(n+1)$-category $\mathcal{C}[\Gamma, \Gamma^-]$, made of $(n+1)$-spheres
$$(\alpha \star_n \alpha^-, 1_{s_n(\alpha)}), \qquad (\alpha^- \star_n \alpha, 1_{t_n(\alpha)}).$$

We refer to [67, 91] for explicit free constructions on cellular extensions over an n-category.

5.4.9. Remark. Given a cellular extension Γ of an n-category $\mathcal{C}$, we denote by Γ^c the set of *cells of* Γ *in context*, that is the set of $(n+1)$-cells of the form

$$f_n \star_{n-1} \ldots \star_2 (f_2 \star_1 (f_1 \star_0 \alpha \star_0 g_1) \star_1 g_2) \star_2 \ldots \star_{n-1} g_n,$$

where f_i, g_i are i-cells of $\mathcal{C}$ for $0 \leq i \leq n$, and $\alpha \in \Gamma$. Recall from [65, Prop. 2.1.5], that any $(n+1)$-cell γ in the free $(n+1)$-category $\mathcal{C}[\Gamma]$ can be written as an n-composition

$$\gamma = \gamma_1 \star_n \gamma_2 \star_n \ldots \star_n \gamma_k,$$

where the γ_i are $(n+1)$-cells of Γ^c, using the algebraic laws of higher categories, most notably the interchange laws. Similarly, any $(n+1)$-cell γ in the free $(n+1,n)$-category $\mathcal{C}(\Gamma)$ can be written as an n-composition

$$\gamma = \gamma_1^{\epsilon_1} \star_n \gamma_2^{\epsilon_2} \star_n \ldots \star_n \gamma_k^{\epsilon_k},$$

where the γ_i are $(n+1)$-cells of Γ^c and $\epsilon_i \in \{-1, 1\}$.

5.4.10. n-polygraphs. Polygraphs generate free higher categories. They are defined by induction on the dimension as follows: for $n \geq 0$, an *n-polygraph* P consists of a set P_0 and for every $0 \leq k < n$ a cellular extension P_{k+1} of the *free k-category*

$$P_0[P_1]\ldots[P_k].$$

For $0 \leq k \leq n$, the elements of P_k are called the *generating k-cells* of P.

The *free n-category* $P_0[P_1]\ldots[P_{n-1}][P_n]$ (*resp.* the *free $(n, n-1)$-category* $P_0[P_1]\ldots[P_{n-1}](P_n)$) *generated by* P will be denoted by P^* (*resp.* $P^\top$).

5.4.11. (n,p)-polygraphs. Just as n-polygraphs generate free n-categories, (n,p)-polygraphs generate free (n,p)-categories. An (n,p) polygraph is a tuple $(P_0, P_1, \ldots, P_n)$ such that

- **i)** $(P_0, P_1, \ldots, P_p)$ is a p-polygraph,
- **ii)** for all $p \leq i < n$, P_{i+1} is a cellular extension of the free (i,p)-category generated by $(P_0, P_1, \ldots P_i)$, denoted by $P_i^\top$. In the notation above, we write
$$P_i^\top = P_p^*(P_{p+1})\ldots(P_k).$$

We refer to [65, 67] for the details on (n,p)-polygraphs.

5.4.12. Underlying rewriting 1-polygraph. Just as in the case of 2-polygraphs, we obtain an underlying 1-polygraph describing the rewriting properties of an n-polygraph. Indeed, it is the final cellular extension that provides the rewriting system on the underlying $(n-1)$-dimensional structure. The latter is the structure presented by an n-polygraph, so its $(n-1)$-cells should constitute the 0-cells of the underlying 1-polygraph. As with 2-polygraphs, it is the n-cells in context which determine the 1-cells.

Explicitly, the *underlying rewriting polygraph* associated to an n-polygraph $P = (P_0, \dots, P_n)$, denoted by P^c, is the 1-polygraph whose 0-cells are given by the set P^*_{n-1} of $(n-1)$-cells in the free $(n-1)$-category generated by $(P_0, \dots, P_{n-1})$, and whose 1-cells are given by P^c_n. Denote by $\tilde{\alpha}$ an element of P^c_n, that is there exist pairs (f_i, g_i) of i-cells, $1 \leq i < n$ and an element $\alpha \in P_n$ such that

$$\tilde{\alpha} = f_{n-1} \star_{n-2} \dots \star_2 (f_2 \star_1 (f_1 \star_0 \alpha \star_0 g_1) \star_1 g_2) \star_2 \dots \star_{n-2} g_{n-1}.$$

The source and target maps of the 1-polygraph P^c are given by those of P^*, that is

$$s(\tilde{\alpha}) = f_{n-1} \star_{n-2} \dots \star_2 (f_2 \star_1 (f_1 \star_0 s_{n-1}(\alpha) \star_0 g_1) \star_1 g_2) \star_2 \dots \star_{n-2} g_{n-1}$$
$$t(\tilde{\alpha}) = f_{n-1} \star_{n-2} \dots \star_2 (f_2 \star_1 (f_1 \star_0 s_{n-1}(\alpha) \star_0 g_1) \star_1 g_2) \star_2 \dots \star_{n-2} g_{n-1}.$$

The underlying rewriting 1-polygraph encodes the properties of P_n as a rewriting system on P^*_{n-1}. Indeed, as a result of Remark 5.4.9, the n-cells of P^* (resp. $P^\top$) are in bijective correspondence with the 1-cells of $(P^c)^*$ (resp. $(P^c)^\top$), as pointed out in Remark 5.4.9.

Therefore, the rewriting properties (confluence, termination, ...) of P are defined to be those of P^c. In particular, we also obtain an underlying ARS (that of P^c) which we denote by $\Rrightarrow_P$. As pointed out in Remark 5.4.9, the correspondence between the relational ARS $\Rrightarrow_{P_n}$ and the categorical point of view holds.

5.4.13. Consistency theorems for n-Polygraphs. We obtain higher dimensional analogs of Theorems 5.1.9 and 5.1.10. These are immediate corollaries of the corresponding results, using the notion of underlying rewriting polygraph and the ARS underlying them.

5.4.14. Corollary (n-Polygraphic Newman's lemma). *Let P be a locally confluent, terminating n-polygraph. Then P is confluent.*

5.4.15. Corollary (n-Polygraphic Church-Rosser theorem). *Let P be an n-polygraph. Then P is confluent if, and only if, P is Church-Rosser.*

We then also obtain an analog of Theorem 5.1.11 as an immediate consequence of the above.

5.4.16. Theorem (n-Polygraphic consistency theorem). *Let P be a locally confluent, terminating n-polygraph. Then P is Church-Rosser.*

5.4.17. Remark. We also have a critical consistency theorem for n-polygraphs, which we do not treat here. Indeed, a notion of critical branchings may also be defined in the case of n-polygraphs, and the proof follows the schema of Theorem 5.3.17; we refer the reader to [65].

This, just as the above theorems do, reflects the independency of notions of abstract or string rewriting from the dimension of the considered polygraph. Indeed, in the case of abstract rewriting, we only need a 1-dimensional structure, hence the introduction of the notion of underlying rewriting 1-polygraph. For string rewriting, we must consider three dimensions: the first two encode the algebraic structure presented by the SRS, and the third represents the rewriting system on it.

Chapter 6.
Algebraic abstract rewriting

In this second preliminary chapter, we present the algebraic approach to rewriting. As described in Section 5.1, systems of calculation find a model in binary relations over a set. The definitions from that section rely on the algebraic structure encoded by the relation algebra over a set. Here, we present a more general description of abstract rewriting in the setting of Kleene algebra, originally introduced as regular algebras by Conway [23]. Kleene algebras have found a variety of applications, notably in language and automata theory [82, 83], as well as in program verification, see [118], also via formal methods [3, 94]. Here, we focus on their use in the description of abstract rewriting systems, notably in [28, 109]. An overview of modal Kleene algebra is provided by [27].

We start in Section 6.1 by providing definitions of the algebraic structures in play, most notably describing how modal operators may be obtained from a formalisation of the notion of (co-)domain. We also provide a novel approach to converses in such modal Kleene algebras in Section 6.1.17, alongside the more classical description, and provide models of Kleene algebra in Sections 6.1.21 and 6.1.22. We then describe how properties of calculation may be expressed in this algebraic context in Section 6.2 from [28]. Notably, we recall Newman's lemma, Theorem 6.2.9, and the Church-Rosser theorem, Theorem 6.2.10, and formulate a consistency theorem for systems of calculation described by Kleene algebra, Theorem 6.2.11.

This chapter presents one new result, first appearing in [17], namely Theorem 6.1.7. Other results and definitions appear in [28, 109].

6.1. Modal Kleene algebras

6.1.1. Semirings. Recall that a *semiring* is a tuple $(S, +, 0, \cdot, 1)$ made of a set S and two binary operations $+$ and $\cdot$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid whose *multiplication operation* $\cdot$ distributes on the left and the right over the *addition operation* $+$, and 0 is a left and right zero for multiplication.

A *dioid* is a semiring S in which addition is idempotent, *i.e.* for all $x \in S$, we have

$x + x = x$. In this case, the relation defined by

$$x \leq y \quad \Longleftrightarrow \quad x + y = y, \tag{6.1.2}$$

for all $x, y \in S$, is a partial order on S, with respect to which addition and multiplication are monotone, and 0 is minimal. Where there is no possible confusion, we will often denote multiplication simply by juxtaposition. A *bounded distributive lattice* is a dioid $(S, +, 0, \cdot, 1)$, whose multiplication $\cdot$ is commutative and idempotent, and $x \leq 1$, for every $x \in S$.

6.1.3. Domain semirings. Recall from [29] that a *domain semiring* is a dioid $(S, +, \cdot, 0, 1)$ equipped with a *domain operation* $d : S \to S$ that satisfies the following five axioms for all $x, y \in S$:

i) $x \leq d(x)x$,

ii) $d(xy) = d(xd(y))$,

iii) $d(x) \leq 1$,

iv) $d(0) = 0$,

v) $d(x + y) = d(x) + d(y)$,

These structures are called domain *semirings* and not domain *dioids* because a semiring equipped with a domain operation is automatically idempotent.

Intuitions for the domain axioms are given in Examples 6.1.21 and 6.1.22 below. In the first, we show that the notion of domain for binary relations satisfies the domain semiring axioms. The second example shows that the algebra of sets of paths over a 1-polygraph satisfies the domain semiring axioms. The domain of a set of paths then corresponds to the set of all sources of paths in the set.

Consequences of the domain semiring axioms include the fact that the image of S under d is the set of fixpoints of d, that is,

$$S_d := \{x \in S \mid d(x) = x\} = d(S),$$

and that S_d forms a distributive lattice with $+$ as join and $\cdot$ as meet, bounded by 0 and 1. It contains the largest Boolean subalgebra of S bounded by 0 and 1 [29]. We henceforth write $p, q, r, \ldots$ for elements of S_d and refer to S_d as the *domain algebra* of S. In particular, S_d is a subsemiring of S in the sense that its elements satisfy the semiring axioms, 0 and 1 are in the set, and the set is closed with respect to $\cdot$ and $+$.

In the relational model of domain semirings, the set S_d consists of the set of all relations included in the identity relation, called *subidentities*. In the path model, it consists of subsets of the set of all paths of length 0. In both cases, the distributive sublattices form Boolean algebras.

Note that by multiplying both sides of axiom **iii)** on the left by x, we have $d(x)x \leq x$. This, combined with axiom **i)** gives

$$d(x)x = x,$$

which coincides with the intuition that restriction of x on the left by its domain does nothing. Further properties of domain semirings include

$$d(0) = 0, \qquad d(px) = pd(x), \qquad x \leq y \Rightarrow d(x) \leq d(y),$$

for all $x, y \in S_d$, and d commutes with all existing sups [29].

6.1.4. Boolean domain semirings. A limitation of domain semirings is that Boolean complementation in S_d cannot be expressed; these structures admit chains as models [29]. Yet complementation is desirable for at least two reasons: it reflects the Boolean nature of the domain algebras of the models in which we are interested. Furthermore, it allows us to define a modal box operator from the modal diamond, built using domain, via standard De Morgan duality, see (6.1.8). We need both Boolean domain algebras and the box-diamond duality in the proof of coherent Newman's lemma in Section 9.3.

To enforce Boolean domain algebras, it is standard to axiomatise a notion of antidomain that abstractly describes those elements that are *not* in the domain of a particular element. The antidomain of a relation, for instance, models the set of all elements that are not related to any other element of the underlying set; the antidomain of a set of paths corresponds to the set of all vertices of the underlying graph that are not a source of any path in the set.

A *Boolean domain semiring* [29] is a dioid $(S, +, 0, \cdot, 1)$ equipped with an *antidomain operation* $ad : S \to S$ that satisfies the following three axioms, for all $x, y \in S$:

i) $ad(x)x = 0$,

ii) $ad(xy) \leq ad(x\, ad^2(y))$,

iii) $ad^2(x) + ad(x) = 1$.

As the antidomain operation is, implicitly, the Boolean complement of the domain operation, the domain of an element is the antidomain of its antidomain, $d = ad^2$. Hence we recover a domain semiring, that is, defining the map d in this way gives a domain semiring. In the presence of the operation ad, the subalgebra S_d of all fixpoints of d in S is now the greatest Boolean algebra in S bounded by 0 and 1, and $S_d = ad(S)$. Finally, ad acts as Boolean complementation on S_d, giving it the structure of a Boolean algebra. We therefore denote the restriction of ad to S_d by $\neg$.

6.1.5. Modal semirings. We denote the *opposite* of a semiring S, in which the order of multiplication has been reversed, by S^{op}. It is once again a semiring.

A *codomain* (resp. *Boolean codomain*) *semiring* is a semiring equipped with a map $r : S \to S$ (resp. $ar : S \to S$) such that (S^{op}, r) (resp. (S^{op}, ar)) is a domain (resp.

Boolean domain) semiring. The codomain operation models the domain of the converse relation in the relational model, and in the path model the set of all targets of paths in a given set of paths.

Consider a semiring equipped with a domain and a codomain operation. The domain and codomain axioms alone do not imply that $S_d = S_r$, let alone the compatibility properties

$$d(r(x)) = r(x), \qquad r(d(x)) = d(x), \tag{6.1.6}$$

for every x in S. Indeed, consider the domain and range semiring $S = (\{a\}, +, 0, \cdot, 1, d, r)$ with addition defined by $0 < a < 1$, multiplication by $a^2 = a$, domain by $d\,a = 1$ and codomain by $r\,a = a$. We have $S_d = \{0, 1\} \neq \{0, a, 1\} = S_r$ and $d\,(r\,a) = 1 \neq a = r\,a$. The identity $r \circ d = d$ fails in the opposite semiring. For this reason, we considering semirings equipped with both a domain and a codomain operation, we impose this compatibility.

A *modal semiring* S [29] is a domain semiring that is also a codomain semiring, and which satisfies the compatibility properties (6.1.6). Boolean domain semirings that are also Boolean codomain semirings are called *Boolean modal semirings.*

In contrast to the non-Boolean case, maximality of S_d and $S_r = \{x \in S \mid r(x) = x\}$ as Boolean subalgebras between 0 and 1 forces the domain and range algebra of S to coincide, so that the extra axioms (6.1.6) are unnecessary.

6.1.7. Theorem ([17]). *In every Boolean modal semiring the compatibility properties* (6.1.6) *hold.*

Proof. Let S be a Boolean modal semiring and x in S. Then

$$\begin{aligned} d(r(x)) &= (ar(x) + r(x))d(r(x)) \\ &= ar(x)d(r(x)) + r(x)d(r(x))(ar(x) + r(x)) \\ &= 0 + r(x)d(r(x))ar(x) + r(x)d(r(x))r(x) \\ &= 0 + r(x)r(x) = r(x), \end{aligned}$$

proving the first equality in (6.1.6). In the third step, $ar(x)d(r(x)) = 0$ because $ar(x)r(x) = 0$ and $yz = 0 \Leftrightarrow yd(z) = 0$ hold in any Boolean modal semiring. In the fourth step, $r(x)d(r(x))ar(x) = 0$ because $d(r(x)) \leq 1$ and again $ar(x)r(x) = 0$. Moreover, $r(x)d(r(x))r(x) = r(x)r(x)$ because $d(y)y = y$ holds in any modal semiring. The proof of the second compatibility property in (6.1.6) follows by opposition. □

In Boolean modal semirings, $d(x) = x$ therefore implies $r(x) = r(d(x)) = d(x) = x$, while $r(x) = x$ implies $d(x) = x$ by opposition. This forces that $S_d = S_r$, as desired.

6.1.8. Modal Operators. In our algebraic approach to higher dimensional rewriting, modal operators are important for relating sets of higher-dimensional cells to their sets of lower-dimensional source and target cells, see (9.1.11), and thus for expressing the forall/exists properties defining fillers and the pasting conditions needed for proofs in higher dimensional rewriting.

As recalled in Section 6.1.21, in the relational model of ARS, given a binary relation R on a set X, we write $|R\rangle P$ to indicate the set of all elements of the underlying set from which the set P *can* be reached following the relation R. We similarly write $\langle R|P$ to denote the set of all elements of X that *can* be reached from P following R. Finally, $|R]P$ indicates the set from which we *must* reach the set P following R and $[R|P$ indicates the set that we *must* reach from P following R. This is consistent with the standard Kripke semantics of forward and backward modal operators in modal and, in particular, dynamic logics. Similar intuitions underlie the path model of modal Kleene algebra, and these generalise to the notions of higher paths and their relations expressed in the filler properties and pasting conditions of higher dimensional rewriting. These explanations motivate the following algebraic definitions of modal operators in modal semirings.

Let $(S, +, \cdot, 0, 1, d, r)$ be a modal semiring. For $x \in S$ and $p \in S_d$, we define the modal diamond operators:

$$|x\rangle p = d(xp), \qquad \langle x|p = r(px). \tag{6.1.9}$$

When S is a Boolean modal semiring, we additionally define modal box operators:

$$|x]p = \neg|x\rangle(\neg p), \qquad [x|p = \neg\langle x|(\neg p). \tag{6.1.10}$$

Beyond the intuitions given, these are modal operators in the sense of Jónsson and Tarski's Boolean algebras with operators [79] because the following identities hold:

$$|x\rangle(p+q) = |x\rangle p + |x\rangle q, \qquad |x\rangle 0 = 0, \qquad \langle x|(p+q) = \langle x|p + \langle x|q, \qquad \langle x|0 = 0,$$

and dually

$$|x](pq) = |x]p + |x]q, \qquad |x]1 = 1, \qquad [x|(pq) = [x|p + [x|q, \qquad [x|1 = 1.$$

It is easy to see that $|-\rangle$ and $\langle -|$, as well as $|-]$ and $[-|$ are related by opposition. In a (Boolean) modal Kleene algebra, following Jónsson and Tarski, this can be expressed by the conjugation laws

$$|x\rangle p \cdot q = 0 \Leftrightarrow p \cdot \langle x|q = 0 \qquad \text{and} \qquad |x]p + q = 1 \Leftrightarrow p + [x|q = 1.$$

In the relational model, it can be expressed explicitly using relational converse.

In a Boolean modal semiring, boxes and diamonds are related by De Morgan duality by their definition (6.1.10) and additionally by

$$|x\rangle p = \neg|x](\neg p) \qquad \text{and} \qquad \langle x|p = \neg[x|(\neg p). \tag{6.1.11}$$

Finally, boxes and diamonds are adjoints in Galois connections:

$$|x\rangle p \le q \Leftrightarrow p \le [x|q \qquad \text{and} \qquad \langle x|p \le q \Leftrightarrow p \le |x]q. \tag{6.1.12}$$

As a consequence, diamonds preserve all existing sups in S, whereas boxes reverse all existing infs to sups, and all modal operators are order preserving. Finally, we mention the properties $|xy\rangle = |x\rangle \circ |y\rangle$, $\langle xy| = \langle y| \circ \langle x|$, $|xy] = |x] \circ |y]$ and $[xy| = [y| \circ [x|$.

6.1.13. Modal Kleene algebras. A *Kleene algebra* is a dioid K equipped with an operation $(-)^* : K \to K$ called *Kleene star*, satisfying the following axioms. For all $x, y, z \in K$,

i) (*unfold axioms*) $1 + xx^* \leq x^*$ and $1 + x^*x \leq x^*$,

ii) (*induction axioms*) $z + xy \leq y \Rightarrow x^*z \leq y$ and $z + yx \leq y \Rightarrow zx^* \leq y$.

Note that the axioms on the left are the opposites of those on the right, in the sense that the order of multiplication has been reversed.

6.1.14. Remark. We take a moment here to explain the intuitions underlying the above axioms. The Kleene star models a finite iteration or repetition of an element x as a least fixpoint. The first unfold axiom, for instance, states that iterating x either amounts to doing nothing, that is, doing 1, or doing x once and then continuing the iteration. Yet while possibly infinite iterations would satisfy such unfold laws, too, the induction laws filter out the least fixpoints to the corresponding pre-fixpoint equations. A more detailed explanation of the induction laws as fixpoints can be found in the literature, see for example [27]. The Kleene star models the reflexive-transitive closure of a relation in the relational model and the repetitive composition of paths in a given set of paths in the path model.

Useful consequences of Axioms **i)** and **ii)** include the following identities for all $x, y \in K$, and $i \in \mathbb{N}$,

$$x^i \leq x^* \quad x^*x^* = x^* \quad x^{**} = x^* \quad x(yx)^* = (xy)x^* \quad (x + y)^* = x^*(yx^*)^* = (x^*y^*)^*,$$

where x^i denotes the i-fold multiplication of x with itself, as well as the quasi-identities

$$x \leq 1 \Rightarrow x^* = 1 \quad x \leq y \Rightarrow x^* \leq y^* \quad xz \leq zy \Rightarrow x^*z \leq zy^* \quad zx \leq yz \Rightarrow zx^* \leq y^*z.$$

The *Kleene plus* is the operation $(-)^+ : K \to K$ defined by $x^+ = xx^*$. It corresponds to the transitive closure in the relational model.

The notions of domain and codomain extend to Kleene algebras without having to add any further axioms describing the interaction of the structures. We thus define a *(Boolean) modal Kleene algebra* as a Kleene algebra whose underlying dioid is a (Boolean) modal semiring.

6.1.15. The algebra of modal operators. In a Boolean modal semiring S, the modal operators obtained via the domain and codomain operations form a Boolean algebra in their own right, denoted by $[\langle S \rangle]$; this is accomplished by point-wise lifting of the operations in the Boolean algebra S_d. For maps $f, g : S_d \to S_d$, called *predicate transformers*, we define

$$\begin{aligned} f \leq g \quad &\Longleftrightarrow \quad \forall p \in B \;\; f(p) \leq g(p), \\ (f + g)(p) &:= f(p) + g(p), \\ (f \sqcap g)(p) &:= f(p)g(p), \\ (\neg f)(p) &:= \neg f(p), \end{aligned}$$

for all $p \in S_d$, where we recall that the multiplication $f(p)g(p)$ is the meet of $f(p)$ and $g(p)$ in B; for the composition of such operators, we will write $f \circ g(p) = f(g(p))$. This defines the algebra of predicate transformers $PT(S_d)$.

Furthermore, the Galois connections between diamonds and boxes lift to the operator level; for all predicate transformers f, g, and $x \in S$,

$$|x\rangle f \leq g \iff f \leq [x|g, \qquad\qquad \langle x|f \leq g \iff f \leq |x]g$$

In the case of an MKA K, the unfold and induction laws are also lifted to operators associated to elements:

$$|1\rangle + |x\rangle|x^*\rangle = |x^*\rangle, \qquad\qquad |1\rangle + |x^*\rangle|x\rangle = |x^*\rangle, \tag{6.1.1}$$

$$|y\rangle + |x\rangle|z\rangle \leq |z\rangle \Rightarrow |x^*\rangle|y\rangle \leq |z\rangle, \tag{6.1.2}$$

$$|x\rangle p + q \leq p \Rightarrow |x^*\rangle q \leq p. \tag{6.1.3}$$

6.1.16. Remark. Recall that a function between f between lattices is *universally disjunctive* (resp. *universally conjunctive*) if f commutes with arbitrary joins (resp. meets), ie suprema (resp. infima). A folklore result from lattice theory states that a function between lattices is universally disjunctive (resp. universally conjunctive) if, and only if, it is a lower (resp. upper) adjoint. The fact that diamonds and boxes are adjoint thus implies that diamonds (resp. boxes) commute with all existing suprema (resp. infima).

Furthermore, since the Galois connection lifts to the operator algebra, we know that composition of diamonds is universally disjunctive in both arguments. Finally, note that by additivity of domain and codomain, we also have that the mapping sending an element $x \in K$ to the corresponding forward or backward diamond (resp. box) is isotone (resp. antitone) from K to $[\langle K\rangle]$.

6.1.17. Converse.. Here we introduce a notion of converse in the setting of modal Kleene algebras. In fact, we consider two different axiomatisations of converse. The first appears in [7] and constitutes the axiomatisation of Kleene algebra with converse found in [10]. The second is inspired by the properties of the path model, and turn out to be similar to concepts present in the theory of inverse semigroups, see for example [85].

A *Kleene algebra with converse* is a Kleene algebra K equipped with an involution $\overline{(-)} : K \to K$, called *converse* or *converse operation*. This map must satisfy the following axioms for all $x, y \in K$:

$$\overline{(x+y)} = \overline{x} + \overline{y}, \qquad \overline{(x \cdot y)} = \overline{y} \cdot \overline{x}, \qquad \overline{(x^*)} = (\overline{x})^*, \qquad \overline{(\overline{x})} = x, \tag{6.1.18}$$

In words, a converse is an additive involution which is contravariant with respect to multiplication and commutes with the Kleene star. To constitute a converse, such an involution must satisfy an additional axiom. We provide two notions of converse corresponding to two different additional axioms.

The first states that for all $x \in K$,

$$x \leq x\overline{x}x. \tag{6.1.19}$$

This axiomatisation is found in [7], but also in [10]. It is independent of the notion of domain, *i.e.* it allows a dioid or Kleene algebra without a domain operation to be equipped with a notion of converse. A converse satisfying axioms (6.1.18) and the additional axiom (6.1.19) will be called a *Gelfand converse.* A similar axiom is found in the theory of inverse semigroups where inverses satisfy $s = ss^{-1}s$, see for example [85].

The second axiomatisation relates conversion to the domain operation. Indeed, it states that converse is contracts onto the domain in the sense that

$$d(x) \leq x\overline{x}, \tag{6.1.20}$$

must hold for all $x \in K$. Again in [85], we have a similar notion, excepting that in that case, the notion of domain and codomain are defined from the inverse operation, *i.e.* $d(s) := ss^{-1}$ and $r(x) := s^{-1}s$. We will call such a converse operation, *i.e.* one satisfying axioms (6.1.18) and (6.1.20), a *contracting converse.*

Note that the first axiom is a consequence of the second. Indeed, suppose that $\overline{}$ is a contracting converse. Multiplying both sides of (6.1.20) on the right by x, we have

$$d(x)x = x\overline{x}x.$$

As explained in Section 6.1.3, we have $d(x)x = x$ as a consequence of domain axioms, so we recover (6.1.19). We observe that in both cases a converse operation exchanges domain and codomain, *i.e.* $d(\overline{x}) = r(x)$ and $r(\overline{x}) = d(x)$. This means that conversion also switches the direction of boxes and diamonds, *i.e.* $|\overline{x}\rangle = \langle x|$ and $|\overline{x}] = [x|$. As a consequence, and using (6.1.12), we have the Galois connections

$$\langle\overline{x}|p \leq q \Leftrightarrow p \leq [x|q \qquad \text{and} \qquad |\overline{x}\rangle p \leq q \quad \Longleftrightarrow \quad p \leq |x]q.$$

Furthermore, conversion is the identity on the domain subalgebra, *i.e.* for $p \in K_d$, $\overline{p} = p$.

Finally, we may also augment Boolean MKA with converse operations. A *(Boolean) MKA with converse*, Gelfand or contracting, is a (Boolean) MKA equipped with such a converse operation.

6.1.21. Example: relation Kleene algebras. Here we put aforementioned intuitions on solid foundations. Binary relations form perhaps the most important model of modal Kleene algebras in program verification applications. The relational model of plain Kleene algebra has been the starting point for Kleene-algebraic proofs of the Church-Rosser theorem of abstract rewriting [109], that of modal Kleene algebra has motivated the Kleene-algebraic proof of Newman's lemma [28].

For any set X, the structure

$$(\mathcal{P}(X \times X), \cup, \emptyset_X, ;, Id_X, (-)^*)$$

forms a Kleene algebra, the *full relation Kleene algebra* over X. This is the algebra $Rel(X)$ described in Section 6.1.21. We recall that the operation ; is relational composition defined by $(a,b) \in R\,;S$ if, and only if, $(a,c) \in R$ and $(c,b) \in S$, for some $c \in X$. The relation $Id_X = \{(a,a) \mid a \in X\}$ is the identity relation on X and $(-)^*$ is the reflexive transitive closure operation defined, for $R^0 = Id_X$ and $R^{i+1} = R\,;R^i$, by

$$R^* = \bigcup_{i\in\mathbb{N}} R^i.$$

The subidentity relations below Id_X form the greatest Boolean subalgebra between $\emptyset_X$ and Id_X, which is isomorphic to the power set algebra $\mathcal{P}(X)$. Every subalgebra of a full relation Kleene algebra is a *relation Kleene algebra*.

The full relation Kleene algebra over X extends to a *full relation Boolean modal Kleene algebra* over X by defining, as expected,

$$d(R) = \{(a,a) \mid \exists b \in X.\ (a,b) \in R\} \qquad \text{and} \qquad r(R) = \{(a,a) \mid \exists b.\ (b,a) \in R\}.$$

The domain algebra $Rel(X)_d$ equals the Boolean algebra of subidentity relations.

The antidomain and anticodomain maps are then given by relative complementation $ad(R) = Id_X \setminus d(R)$ and $ar(R) = Id_X \setminus r(R)$ within the domain algebra. Finally, it is straightforward to check that the algebraic definitions of boxes and diamonds expand to their standard relational Kripke semantics:

$$|R\rangle P = \{(a,a) \mid \exists b \in X.\ (a,b) \in R \wedge (b,b) \in P\},$$
$$|R]P = \{(a,a) \mid \forall b \in X.\ (a,b) \in R \Rightarrow (b,b) \in P\},$$

and likewise for the backward modalities. This requires swapping (a,b) to (b,a) in the above expressions, which amounts to taking relational converse.

6.1.22. Example: path Kleene algebras. The path model of modal Kleene algebras is a stepping stone towards polygraph models of higher Kleene algebras. Instead of a 1-polygraph, we could speak of a directed graph or quiver. So let P^* be the free 1-category generated by the 1-polygraph $P = (P_0, P_1)$. Its elements are paths in P to which we assign source and target maps s_0 and t_0 as well as a path composition $\star_0$ as a pullback of s_0 and t_0 in the standard way. Then $(\mathcal{P}(P_1^*), \cup, \emptyset, \odot, \mathbb{1}, (-)^*)$ forms a Kleene algebra, the *full path (Kleene) algebra* $K(P)$ over P. Here, composition is defined as a complex product

$$\phi \odot \psi = \{\ u \star_0 v \ \mid\ u \in \phi\ \wedge\ v \in \psi\ \wedge\ t_0(u) = s_0(v)\ \}$$

for any $\phi, \psi \in \mathcal{P}(P_1^*)$, and $\mathbb{1}$ is the set of all identity arrows, or paths of length zero, of P. The Kleene star is defined as

$$\phi^* = \bigcup_{i\in\mathbb{N}} \phi^i$$

where $\phi^0 = \mathbb{1}$ and $\phi^{i+1} = \phi \odot \phi^i$. It models the repetitive composition of the paths in ϕ mentioned before. In particular, notice that P_1 is an element of $\mathcal{P}(P_1^*)$ Every subalgebra of the full path Kleene algebra over P is a *path Kleene algebra.* As in the case of relational Kleene algebras, the set of all subidentities (subsets of $\mathbb{1}$), the set of sets of identity arrows, forms a Boolean subalgebra.

The full path algebra over P extends to a *full path Boolean modal Kleene algebra* over P by defining

$$d(\phi) = \{1_{s(u)} \mid u \in \phi\} \qquad \text{and} \qquad r(\phi) = \{1_{t(u)} \mid u \in \phi\}$$

where 1_x denotes the identity arrow on an object $x \in P_0$. The domain algebra induced equals the Boolean algebra of subidentities. The antidomain and anticodomain maps are therefore given again by relative complementation $ad(\phi) = \mathbb{1} \setminus d(\phi)$ and $ar(\phi) = \mathbb{1} \setminus r(\phi)$ within the domain algebra. Finally, unfolding definitions shows that

$$|\phi\rangle p = \{1_{s(u)} \mid u \in \phi \wedge t(u) \in p\} \qquad \text{and} \qquad |\phi]p = \{1_{s(u)} \mid u \in \phi \Rightarrow t(u) \in p\},$$

where $p \subseteq \mathbb{1}$ is some set of identity arrows. Reachability along a relation has now been replaced by reachability along a set of paths. Similar expressions for backward modalities can be obtained again by swapping source and target maps in the right places. As a final remark, note that $|P_1\rangle$ corresponds to the (forward) modal diamond operator for the underlying ARS $\rightarrow_P$ associated to ϕ.

The relational model and the path model are very similar. In fact the relational model can be obtained from the path model by applying a suitable homomorphism of modal Kleene algebras.

6.2. ABSTRACT REWRITING IN MODAL KLEENE ALGEBRAS

The mechanisms of abstract rewriting may be expressed succinctly in the abstract setting of Kleene algebra. In Sections 5.1 and 5.2, we recalled the relational and polygraphic approaches to abstract rewriting theory. The models of modal Kleene algebra above, (6.1.21) and (6.1.22), show that both of these settings fall under the scope of Kleene algebra. The rewriting properties described in the following paragraphs correspond to those introduced in the case of relations or 1-polygraphs when interpreted in the models. This is made precise in Section 9.5.

6.2.1. Termination and normal forms in MKA.

We recall from [28] definitions and results concerning termination and normal forms expressed in MKA.

i) An element $x \in K$ *terminates*, or is *Noetherian*, provided that for all $p \in K_d$ the implication

$$p \leq |x\rangle p \Rightarrow p = 0,$$

holds. Recall that the definition of termination for ARS given in Section 5.1.4 is precisely this condition in the relational model of Kleene algebra. The set of

Noetherian elements of K is denoted by $\mathcal{N}(K)$. The Galois connections (6.1.12) yield the following equivalent characterisation of termination:

$$\forall p \in K_d, \quad |x]p \leq p \Rightarrow p = 1. \tag{6.2.2}$$

ii) The *exhaustion* of an element $x \in K$, denoted by $exh(x)$, is defined by

$$exh(x) := x^* \cdot \neg d(x). \tag{6.2.3}$$

iii) The *normal forms element* of $x \in K$, denoted by $\mathrm{NF}(x)$, is defined by

$$\mathrm{NF}(x) := r(exh(x)) \in K_d. \tag{6.2.4}$$

We will show in the following subsection how these notions correspond to those introduced in Sections 5.1 and 5.2 when instantiated in the corresponding models. Before finishing this subsection, we recall a lemma stating that if x is a terminating element of K, then a normal form may be reached from any element.

6.2.5. Lemma ([28]). *Let K be a Boolean modal Kleene algebra and $x \in K$. If x terminates, then $d(exh(x)) = 1$.*

6.2.6. Confluence properties in MKA. Confluence properties are captured in MKA by semi-commutation properties. We distinguish two cases of semi-commutation: (regular) semi-commutation and modal semi-commutation, the latter being less fine. Given $x, y \in K$, we say that the ordered pair (x, y)

i) *semi-commutes locally* if $xy \leq y^*x^*$,

ii) *semi-commutes* if $x^*y^* \leq y^*x^*$, and

iii) has the *Church-Rosser property* if $(x + y)^* \leq y^*x^*$.

Rather than expressing confluence and Church-Rosser properties as we have seen thus far, these properties can be thought of as expressing a *reordering* of calculation steps. Indeed, without a notion of converse, nor using the notion of modalities, see below, we cannot express the notion of branching or confluence as seen in abstract rewriting.

As examples, the semi-commutation of the pair (x, y) expresses that any elements related by an iteration of x-steps followed by and iteration of y-steps, may also be related by an iteration of y-steps followed by an iteration of x-steps. In this sense, the Church-Rosser property for MKA states a re-ordering property: given elements related by steps in x and y in any order, we may relate them using y-steps followed by x-steps.

To relate these notions to diagrammatic intuitions, we will represent these situations by the following diagrams

$$\begin{tikzcd}
 & \cdot \arrow[dr, "y"] & \\
\cdot \arrow[ur, "x"] \arrow[dr, "y^*"'] & & \cdot \\
 & \cdot \arrow[ur, "x^*"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \cdot \arrow[dr, "y^*"] & \\
\cdot \arrow[ur, "x^*"] \arrow[dr, "y^*"'] & & \cdot \\
 & \cdot \arrow[ur, "x^*"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[rr, leftrightarrow, "(x+y)^*"] \arrow[dr, "y^*"'] & & \cdot \\
 & \cdot \arrow[ur, "x^*"'] &
\end{tikzcd}$$

where, from left to right, we have represented the local semi-commutation, the semi-commutation, and the Church-Rosser property of the pair (x, y).

In the presence of a notion of converse, Gelfand or contracting, we may describe confluence and Church-Rosser properties of a single element. These correspond to their homonyms in the context of ARS. Let $\overline{(-)}$ be a converse operation on K. We say that an element $x \in K$ is (*locally*) *confluent* (resp. *Church-Rosser*) if the pair $(\overline{x}, x)$ semi-commutes (resp. has the Church-Rosser property). Finally, we say that x is *convergent* if it is both terminating and confluent.

Now we turn to modal semi-commutation properties. Given $x, y \in K$, we say that the ordered pair (x, y)

- **i)** *modally semi-commutes locally* if $\langle x||y\rangle \leq |y^*\rangle\langle x^*|$,
- **ii)** *modally semi-commutes* if $\langle x^*||y^*\rangle \leq |y^*\rangle\langle x^*|$, and
- **iii)** has the *modal Church-Rosser property* if $|(x+y)^*\rangle \leq |y^*\rangle\langle x^*|$.

Notice that in the presence of a converse operation, these properties are consequences of the corresponding (regular) properties listed above. Indeed, applying the diamond operator to both sides of the above inequalities in the case of the ordered pair $(\overline{x}, y)$, we obtain the corresponding modal property for the pair (x, y). In this way, using forward and backward diamonds models a converse operation under the modality. For this reason, we represent these situations graphically by

$$\begin{tikzcd}
& \cdot \arrow[dl, "x"'] \arrow[dr, "y"] & \\
\cdot \arrow[dr, "y^*"'] & & \cdot \arrow[dl, "x^*"] \\
& \cdot &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \cdot \arrow[dl, "x^*"'] \arrow[dr, "y^*"] & \\
\cdot \arrow[dr, "y^*"'] & & \cdot \arrow[dl, "x^*"] \\
& \cdot &
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[rr, leftrightarrow, "(x+y)^*"] \arrow[dr, "y^*"'] & & \cdot \arrow[dl, "x^*"] \\
& \cdot &
\end{tikzcd}$$

As above, we will say that an element $x \in K$ is (*locally*) *modally confluent* (resp. *modally Church-Rosser*) if the pair $(\overline{x}, x)$ modally semi-commutes (resp. has the modal Church-Rosser property). Finally, we say that x is *modally convergent* if it is both terminating and modally confluent.

As in the case of ARS and 1-polygraphs, we have a result relating confluence to unicity of normal forms.

6.2.7. Lemma ([28]). *Let K be a Boolean modal Kleene algebra and $x \in K$. If x is (modally) confluent, then $exh(x)$ is deterministic,* i.e. $\langle exh(x)|\ |exh(x)\rangle \leq \langle 1\rangle$.

6.2.8. Consistency theorem in MKA. Just as in the case of ARS and 1-polygraphs, we obtain a consistency theorem for elements of an MKA. As usual, we first provide statements of Newman's lemma and the Church-Rosser theorem for MKA. The consistency theorem is a direct consequence.

In the case of Newman's lemma, we prove the result relative to modal semi-commutation properties.

6.2.9. Theorem (Newman's Lemma [28]). *Let (K, B) be a modal Kleene algebra with K_d a complete Boolean algebra, and let $x, y \in K$ such that $(x + y) \in \mathcal{N}(K)$.*

If (x, y) locally modally semi-commutes, then (x, y) modally commutes.

Proof. We first define a predicate $rc(-)$ which states that x and y modally commute up to restriction to a domain element $p \in K_d$:

$$rc(p) \quad \Longleftrightarrow \quad \langle x^*|\langle p\rangle|y^*\rangle \leq |y^*\rangle\langle x^*|$$

The operator $\langle x^*|\langle p\rangle|y^*\rangle$ sends a domain element q to $r(x^*(pd(y^*q)))$, which, in the relational intuition, corresponds to the following situations:

$$u \xleftarrow{x^*} w \xrightarrow{y^*} v,$$

with $v \in q$, $w \in p$ and u in the image of the operator. It thereby corresponds to branchings of y^* and x^* on points of p.

Note that by completeness of K_d, the supremum $r := sup\{p \in K_d \mid rc(p)\}$ exists. Furthermore, by Remark 6.1.16, $\langle r\rangle = sup\{\langle p\rangle \mid rc(p)\}$, and since composition of diamonds is universally disjunctive (*i.e.* commutes with suprema), we may infer $rc(r)$.

We proceed in two steps; the first is a proof that the induction step predicate, defined for $p \in K_d$ by

$$Ind(p) \quad \Longleftrightarrow \quad rc(p_y) \wedge rc(p_x) \Rightarrow rc(p),$$

holds for all $p \in K_d$, where $p_z := \langle z|p$ for $z \in K$. The second step shows that the statement $(\forall p, Ind(p))$ is equivalent to the retraction of r by $|a+b]$, *i.e.* that $|a+b]r \leq r$. By the box characterization of termination (6.2.2), this will conclude the proof, since then we must have $r = 1$.

Firstly, note that *codomain propagation*, *i.e.*

$$\langle p\rangle|z\rangle = |pz\rangle = |pz\rangle\langle p_z\rangle \leq |z\rangle\langle p_z\rangle$$

holds for all $p \in K_d$ and $z \in K$ since $z = zr(z)$, $p, p_x \leq 1$, and because the domain operator is isotone. Similarly, $\langle z|\langle p\rangle \leq \langle p_z\rangle\langle z|$ holds for all $p \in K_d$ and $z \in K$.

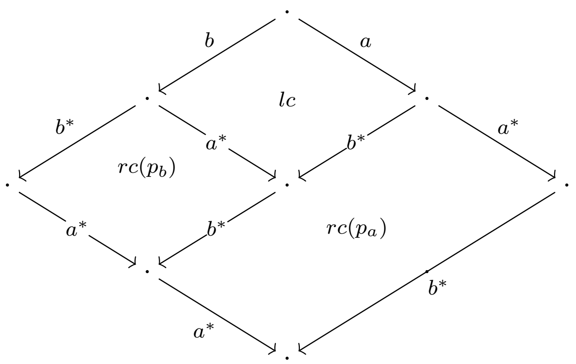
Next, we use the star unfold law (6.1.1) and the reflexivity of the Kleene star, *i.e.* $1 \leq z^*$ for $z \in K$, to bound the branchings of y and x at p:

$$\begin{aligned}
\langle x^*|\langle p\rangle|y^*\rangle &= (\langle 1| + \langle x^*|\langle x|)\langle p\rangle|y^*\rangle \\
&= \langle p\rangle|y^*\rangle + \langle x^*|\langle x|\langle p\rangle(|1\rangle + |y\rangle|y^*\rangle) \\
&= \langle x^*|\langle x|\langle p\rangle + \langle x^*|\langle x|\langle p\rangle|y\rangle|y^*\rangle + \langle p\rangle|y^*\rangle \\
&\leq \langle x^*|\langle p\rangle + \langle x^*|\langle x|\langle p\rangle|y\rangle|y^*\rangle + \langle p\rangle|y^*\rangle.
\end{aligned}$$

In this way we split iterations of x and y into single steps followed by iteration. We have $\langle x^*|\langle p\rangle, \langle p\rangle|y^*\rangle \leq |y^*\rangle\langle x^*|$, again by reflexivity, so we only need to bound the middle summand to conclude the first part of the proof.

Bounding the middle summand is accomplished by the following calculation, in which we use the hypothesis that (x, y) commutes locally, codomain propagation, $rc(p_x)$ followed by idempotence of multiplying the Kleene star of a given element, *i.e.* $x^*x^* = x^*$, and then the same for $rc(p_y)$. The calculation is illustrated on the right in terms of the diagrammatic proof:

$$\begin{aligned}
\langle x^*|\langle x|\langle p\rangle|y\rangle|y^*\rangle &\leq \langle x^*|\langle p_x\rangle\langle x||y\rangle\langle p_y\rangle|y^*\rangle \\
&\leq \langle x^*|\langle p_x\rangle|y^*\rangle\langle x^*|\langle p_y\rangle|y^*\rangle \\
&\leq |y^*\rangle\langle x^*|\langle x^*|\langle p_y\rangle|y^*\rangle \\
&\leq |y^*\rangle\langle x^*|\langle p_y\rangle|y^*\rangle \\
&\leq |y^*\rangle|y^*\rangle\langle x^*| \\
&\leq |y^*\rangle\langle x^*|
\end{aligned}$$



We have thus proved $(\forall p \in K_d, Ind(p))$, concluding the first part of the proof.

To finish, we show that the above statement is equivalent to retraction of r by $|x + y]$. Since the set $\{p \in K_d \mid rc(p)\}$ is downward closed and by definition of suprema, we have $p \leq r \iff rc(p)$. This extends to the induction hypothesis:

$$\begin{aligned}
\forall p\ (rc(p_y) \wedge rc(p_x) \Rightarrow rc(p)) \quad &\iff \quad \forall p\ (\ (p_y = \langle y|p \leq r\ )\ \wedge\ (\ p_x = \langle x|p \leq r\ ) \Rightarrow p \leq r) \\
&\iff \quad \forall p\ (\ \langle y + x|p \leq r\ \Rightarrow p \leq r),
\end{aligned}$$

where the second equivalence is by definition of meet and by additivity of the diamond operator. By the Galois connection, the above is equivalent to

$$\forall p\ (\ p \leq |x + y]r\ \Rightarrow p \leq r),$$

which, by definition of suprema, is equivalent to $|x + y]r \leq r$, thus concluding the proof. Indeed, by termination, this implies that $r = 1$, meaning that x and y modally semi-commute. □

As illustrated by the proof, the reason that we only have a modal version of this theorem in the setting of MKA is for the use of abstract Noetherian induction, the notion of termination being modally defined.

For the Church-Rosser theorem, we have results both in terms of regular and modal semi-commutation properties:

6.2.10. Theorem ((modal) Church-Rosser for MKA [109]). *Let K be a (modal) Kleene algebra and $x, y \in K$. Then (x, y) (modally) semi-commutes if, and only if, (x, y) has the (modal) Church-Rosser property.*

As a direct consequence of the above, we obtain a modal consistency theorem for MKA.

6.2.11. Theorem (Consistency for MKA). *Let K be a modal Kleene algebra and $x, y \in K$ such that $(x + y) \in \mathcal{N}(K)$ and (x, y) locally modally semi-commute. Then (x, y) has the Church-Rosser property.*

Applying the above theorems to a single element of K, we recover the corresponding theorems for ARS, see Section 9.5

CHAPTER 7.
COHERENCE VIA REWRITING

In this third preliminary chapter, we recall the notion of *coherence*. This concept is central to this thesis. In a higher categorical structure, certain algebraic properties, *e.g.* associativity of composition, may only hold up to the existence of higher-dimensional morphisms. Given a collection of such higher morphisms, *coherence* is the requirement that the whole structure is contractible, *i.e.* all parallel morphisms are linked by higher morphisms. A *coherence theorem* states that, given a (generating) collection of such morphisms, coherence is satisfied.

The use of rewriting in questions of coherence was initiated by Squier in [104]. The main point is to compute extensions of an algebraic structure by *homotopy generators* which take the relations amongst the rewriting paths into account. This rewriting method for coherence was applied to solve coherence problems in algebra [25, 46, 69], and for monoidal categories [66]. There, the higher cells constitute a truncations of a cofibrant replacement of the monoid presented by the SRS [46, 67]. A general description of higher dimensional rewriting paradigms may be found in [1].

While coherence proofs by rewriting are similar to consistency proofs, the key difference is that the homotopy generators, although encoded in the familiar structure of higher polygraphs, are not considered as a rewriting system, but as an equivalence relation on (higher) rewriting paths.

We start in Section 7.1 by introducing the structures used to describe coherence of ARS, namely $(2, 0)$-polygraphs, and define homotopy bases and coherent rewriting. We present different versions of Newman's lemma and the Church-Rosser theorem, first proving coherent versions of the classical theorems, then normalising versions thereof. We then prove the coherence theorem for ARS using two different techniques, first without and then with the notion of strategy. The former approach is an original contribution, achieving a proof of the coherence theorem using the aforementioned classical results, while the latter involves strategic versions thereof. Sections 7.2 and 7.3 extend these notions to the string rewriting and higher dimensional rewriting paradigms, respectively.

This chapter presents no explicitly new material, although the first proof of Theorem 7.1.16, which does not use the notion of strategy, is not in the literature in polygraphic language. Results from this section may be found in [65, 68].

7.1. COHERENCE FOR ABSTRACT REWRITING SYSTEMS

We first treat the case of 1-polygraphs. In Section 5.3, we defined 2-polygraphs as cellular extensions of the free category generated by a 1-polygraph as a means of describing rewriting systems on algebraic structures, *i.e.* string rewriting systems. Here we will again consider 2-dimensional structures, but with a different approach, defining coherence of a 1-polygraph with respect to a cellular extension of the free groupoid it generates. In this sense, we study the coherence properties of the equivalence generated by the 1-polygraph in question.

7.1.1. $(2,0)$-polygraphs. Let $P = (P_0, P_1)$ be a 1-polygraph, and consider a cellular extension Γ of the free groupoid $P^\top$. In the terminology of Section 5.4.11, we say that the triple (P_0, P_1, Γ) is a $(2,0)$-polygraph. This generates the free 2-groupoid $P^\top(\Gamma)$, also denoted $\Gamma^\top$ to simplify notation, in which 0-cells are elements of P_0, 1-cells are zig-zag sequences of P *i.e.* elements of $P_1^\top$, and 2-cells are formal composites of elements of Γ and their formal inverses. As defined in Section 5.3, this 2-groupoid is the quotient 2-category

$$\Gamma^\top = P^\top[\Gamma, \Gamma^-]/Inv(\Gamma).$$

A $(2,0)$ polygraph (P_0, P_1, Γ) *presents* a 1-groupoid $\overline{P}^\top$, given by the quotient $P^\top/\Gamma$.

7.1.2. Homotopy bases and coherence. Let (P_0, P_1, Γ) be a $(2,0)$-polygraph. We say that Γ is a *homotopy basis* if for all parallel zig-zag sequences f and g in P, *i.e.* those with common source and target, there exists a 2-cell α in $P^\top(\Gamma)$ such that $s_1(\alpha) = f$ and $t_1(\alpha) = g$. Symbolically and diagrammatically stated:

$$\forall f, g \in P_1^\top, \begin{cases} s_0(f) = s_0(g) \\ t_0(f) = t_0(g) \end{cases} \quad \exists \alpha \in P^\top(\Gamma)_2, \alpha : f \Rightarrow g.$$

$$\begin{tikzcd}
\cdot \arrow[r, bend left, leftrightarrow, "f"] \arrow[r, bend right, leftrightarrow, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & \cdot
\end{tikzcd}$$

Note that this is equivalent to saying that the the 1-groupoid presented by (P_0, P_1, Γ) is acyclic. A homotopy basis is also called a *coherent extension* of P.

7.1.3. Coherent abstract rewriting. We now define rewriting properties of an ARS $P = (P_0, P_1)$ in terms of a cellular extension Γ of $P^\top$.

i) We say that a branching (resp. local branching) (f, g) of P is Γ-*confluent* provided that there exists a confluence (f', g') and a 2-cell $\alpha \in \Gamma^\top$ such that $\alpha : ff' \Rightarrow gg'$, as in the following diagram:

$$\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x_1 \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, "f'"'] & & x_2 \arrow[dl, "g'"] \\
 & x' &
\end{tikzcd}$$

We say that P is Γ-*confluent* (resp. *locally* Γ-*confluent*) if every branching (resp. local confluence) is Γ-confluent.

ii) We say that a branching (f, g) of P is Γ-*normalised* if it is Γ-confluent and the target of the confluence (f', g') is a normal form, as pictured in the following diagram

$$\begin{tikzcd}
& x \arrow[ld, "f"'] \arrow[rd, "g"] & \\
x_1 \arrow[rr, Rightarrow, "\alpha"] \arrow[rd, "f'"'] & & x_2 \arrow[ld, "g'"] \\
& \hat{x} &
\end{tikzcd}$$

where $\hat{x}$ is a normal form of P. We say that P is *normally* Γ-*confluent* when every branching of P is Γ-normalised.

iii) A zig-zag sequence $h : u \leftrightarrow v$ of P is Γ-*confluent* provided that there exist rewriting sequences k and k' of P and a 2-cell $\alpha \in \Gamma^{\top}$ such that $\alpha : k \Rightarrow hk'$, as in the following diagram:

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h"] \arrow[rd, "k"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"', shift right=3] & & v \arrow[ld, "k'"] \\
& u' &
\end{tikzcd}$$

We say that P is Γ-*Church-Rosser* if every zig-zag sequence of P is Γ-confluent.

iv) A zig-zag sequence $h : u \leftrightarrow v$ of P is Γ-*normalised* provided that there exist normalising rewriting sequences k and k' of P and a 2-cell $\alpha \in \Gamma^{\top}$ as in the following diagram:

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h"] \arrow[rd, "k"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"', shift right=3] & & v \arrow[ld, "k'"] \\
& \hat{u} = \hat{v} &
\end{tikzcd}$$

We say that P is *normally* Γ-*Church-Rosser* if every zig-zag sequence of P is Γ-normalised.

v) Two parallel zig-zag sequences f and g are Γ-*congruent* if there exists a 2-cell $\alpha \in \Gamma^{\top}$ such that $\alpha : f \Rightarrow g$. Γ is then a homotopy basis when all parallel zig-zags are Γ-congruent.

We obtain coherent versions of the fundamental theorems of abstract rewriting, recalled in Section 5.2.

7.1.4. Theorem (Coherent Church-Rosser theorem). *Let P be a 1-polygraph, and Γ be a cellular extension of $P^{\top}$. Then P is Γ-confluent if, and only if, P is Γ-Church-Rosser.*

Proof. Let $h : u \leftrightarrow v$ be a zig-zag sequence, *i.e.* a 1-cell of $P^{\top}$. We proceed by induction on the length of h. If h is an identity, that is if $u = v$ and $h = 1_u$, the identity 2-cell 1_{1_u} suffices to prove Γ-confluence. Suppose now that h is of length $k \geq 1$. We write $h = h_1 \star_0 h_2$, where $h_1 : u \leftrightarrow w$ is a zig-zag sequence of length $(k - 1)$ and, without loss of generality, h_2 is either a rewriting step or the formal inverse of one, *i.e.* either $h_2 : w \to v$ or $h_2^- : v \to w$. In the first case, we obtain a diagram of the form

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h_1"] \arrow[dr, "k_1"'] & & w \arrow[rr, "h_2"] \arrow[dl, "k_2"] & & v \arrow[dl, "k'"] \\
& u' \arrow[rr, "k"'] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & & v' \arrow[ur, phantom, "\overset{\beta}{\Longrightarrow}"] &
\end{tikzcd} \tag{7.1.5}$$

where α is obtained by the induction hypothesis and β by the hypothesis of Γ-confluence. The composite $(\alpha \star_0 k) \star_1 (h_1 \star_0 \beta)$ makes the zig-zag h Γ-confluent.

In the second case, we obtain a diagram of the form

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h_1"] \arrow[dr, "k_1"'] & & w \arrow[rr, "h_2^-"] \arrow[dl, "k_2"] & & v \\
& u' \arrow[urrr, bend right, "k_2^- h_2^-"'] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & & = &
\end{tikzcd} \tag{7.1.6}$$

The 2-cell $\alpha h_2^- k_2^-$ makes the zig-zag h Γ-confluent. $\square$

7.1.7. Theorem (Coherent Newman lemma). *Let P be a terminating 1-polygraph, and Γ a cellular extension of $P^\top$. If P is locally Γ-confluent, then P is Γ-confluent.*

Proof. Let u be a 0-cell of P. We proceed by Noetherian induction, proving that every branching with source u is Γ-confluent. If u is a normal form, the only branchings on u are of the form $(1_u, 1_u)$, so the identity 2-cell 1_{1_u} suffices to prove Γ-confluence.

For the induction step, suppose now that u is reducible and that Γ is a confluence filler for any branching with source u' such that there exists a reduction sequence $u \to u'$, and let (f, g) be a branching of source u. If f or g is an identity, the identity 1_g or 1_f, respectively, suffices. If not, we write $f = f_1 \star_0 f_2$ and $g = g_1 \star_0 g_2$ where f_1, g_1 are rewriting steps and f_2, g_2 are rewriting sequences. By the local Γ-confluence hypothesis, we obtain a 2-cell alpha as in the diagram (7.1.8) below. By the induction hypothesis on u_1 and (f_2, f_1'), we obtain the 2-cell β, and subsequently applying the induction hypothesis to v_1 and $(g_1' h, g_2)$, we obtain the 2-cell γ as pictured below.

$$\begin{tikzcd}
& & u \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & \\
& u_1 \arrow[dl, "f_2"'] \arrow[dr, "f_1'"] & \overset{\alpha}{\Longrightarrow} & v_1 \arrow[dl, "g_1'"'] \arrow[dr, "g_2"] & \\
u_2 \arrow[dr, "f_2'"'] & \overset{\beta}{\Longrightarrow} & u' \arrow[dl, "h"] & \overset{\gamma}{\Longrightarrow} & v_2 \arrow[ddll, "g_2'"] \\
& u_2' \arrow[dr, "k"'] & & & \\
& & u'' & &
\end{tikzcd} \tag{7.1.8}$$

Then the following composite

$$\delta = (((f_1 \star_0 \beta) \star_1 (\alpha \star_0 h)) \star_0 k) \star_1 (g_1 \star_0 \gamma) \tag{7.1.9}$$

is a 2-cell in $P^*(\Gamma)$ with source $f \star_0 (f_2' \star_0 k)$ and target $g \star_0 g_2'$, proving the result. □

Note that for $\Gamma = \mathrm{Sph}(P^*)$, Theorems 7.1.4 and 7.1.7 correspond to Newman's lemma [92] and the Church-Rosser theorem [22], recalled in Section 5.4 as Corollaries 5.4.14 and 5.4.15, see also [76].

Note that as corollaries, we obtain normalising versions of Theorems 7.1.4 and 7.1.7:

7.1.10. Corollary (Coherent normalising Church-Rosser theorem). *Let P be a 1-polygraph, and Γ a cellular extension of $P^\top$. Then P is normally Γ-confluent, if, and only if, P is normally Γ-Church-Rosser.*

Proof. The proof is identical to the proof of Theorem 7.1.4, but with normalising confluences. □

7.1.11. Corollary (Coherent normalising Newman lemma). *Let P be a terminating 1-polygraph, and Γ a cellular extension of $P^\top$. If P is locally Γ-confluent, then P is Γ-normalising.*

Proof. The proof is identical to the proof of Theorem 7.1.7, except for that when we choose the confluences (f_2', h) and (k, g_2'), we choose normalising reductions. Note that in this case, k is an identity and $u_2' = u'' = \hat{u}$. More precisely, given a 0-cell u, we prove by Noetherian induction that any branching with source u is Γ-normalised. □

7.1.12. Family of generating confluences. Let P be a locally confluent 1-polygraph. A *family of generating confluences* for P is a cellular extension Γ consisting of a 2-cell $\alpha_{f,g}$ for every local branching (f, g) and the choice of a confluence (f', g') as in the following diagram:

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x_1 \arrow[dr, "f'"'] \arrow[rr, Rightarrow, "\alpha_{f,g}"] & & x_2 \arrow[dl, "g'"] \\
& x' &
\end{tikzcd}$$

In the next subsection, we will see that such a cellular extension is a coherent extension when P is convergent.

7.1.13. Squier's theorem for ARS. Coherence for a convergent 1-polygraph with respect to a family of generating confluences is proved in two steps: first, we show that for any parallel normalising reduction sequences (f, g), there exists a 2-cell $\alpha : f \Rightarrow g$ in $\Gamma^\top$. Second, we apply the Corollary 7.1.10 and 7.1.11 to parallel zig-zags.

The proof of the first step is very similar to that of Corollary 7.1.10, but instead of quantifying over branchings, we are quantifying over parallel normalising reductions.

This distinction will be important when we formulate these theorems in higher Kleene algebras.

7.1.14. Proposition. *Let P be a locally confluent, terminating 1-polygraph and Γ a family of generating confluences. Then for any two normalising reduction sequences $f, g : u \to \hat{u}$, there exists a 2-cell $\alpha : f \Rightarrow g$ in $\Gamma^\top$.*

Proof. This proof is again by Noetherian induction on the source u of f and g. For the base case, as in the proof of Theorem 7.1.7, it suffices to take the 2-cell 1_{1_u}.

For the induction step, we suppose that the property holds for normalising reduction sequences with source u' such that there exists a reduction sequence $u \to u'$. Since $\hat{u}$ is a normal form, we don't need to consider the case in which f or g is an identity. We again write $f = f_1 \star_0 f_2$ and $g = g_1 \star_0 g_2$ as in the case of Theorem 7.1.7. By the local Γ-confluence hypothesis, we obtain a 2-cell α of Γ as in diagram (7.1.15). Applying the induction hypothesis to the the parallel normalising reductions with source u_1 and u_2, we obtain 2-cells β and γ of $\Gamma^\top$ as in the below diagram.

$$\begin{tikzcd}
& u \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
u_1 \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[dr, dashed, "f_1'"] \arrow[ddr, bend right, "f_2"'] & & u_2 \arrow[dl, dashed, "g_1'"'] \arrow[ddl, bend left, "g_2"] \\
\overset{\beta}{\Longrightarrow} & u' \arrow[d, "k"] & \overset{\gamma}{\Longrightarrow} \\
& \hat{u} &
\end{tikzcd} \tag{7.1.15}$$

The composite 2-cell $(f_1\beta) \star_1 (\alpha k) \star_1 (g_1\gamma)$ completes the proof. □

This leads to the first proof of the coherence theorem for 1-polygraphs, formulated in [65], see also [1].

7.1.16. Theorem (Coherence theorem for 1-polygraphs). *Let P be a locally confluent, terminating 1-polygraph and Γ a family of generating confluences. Then Γ is a homotopy basis for P.*

We provide two proofs of the theorem. The first is an original proof not using rewriting strategies, while the second is the more classical polygraphic proof from [65].

Proof. Firstly, by Theorem 7.1.11, we know that P is normally Γ-confluent, and therefore, by Theorem 7.1.10, it is normally Γ-Church-Rosser. Furthermore, by Proposition 7.1.14, we know that any two parallel, normalising reduction sequences are Γ-congruent. Given parallel zig-zag sequences $h_1, h_2 : u \leftrightarrow v$, we may therefore build the following diagram, where $\alpha_1 : k_1 \Rightarrow h_1 k_1'$ and $\alpha_2 : k_2 \Rightarrow h_2 k_2'$:

$$\begin{tikzcd}
u \arrow[rrrr, leftrightarrow, bend left=60, "h_1"] \arrow[rrrr, leftrightarrow, bend right=60, "h_2"'] \arrow[rrrr, phantom, bend left=35, "\xRightarrow{\alpha_1}"] \arrow[rrrr, phantom, bend right=35, "\xRightarrow{\alpha_2}"] \arrow[rr, bend left, "k_1"] \arrow[rr, bend right, "k_2"'] & \Downarrow\beta_1 & \hat{u} = \hat{v} & \beta_2\Downarrow & v \arrow[ll, bend right, "k_1'"'] \arrow[ll, bend left, "k_2'"]
\end{tikzcd} \tag{7.1.17}$$

Recall from Section 5.4.7 that whiskering the inverse of β_2 by the inverses of its source and target provide a 2-cell $(k_1')^- \beta_2^- (k_2')^- : (k_1')^- \Rightarrow (k_2')^-$. Similarly, $\alpha_1^-(k_1')^- : h_1 \Rightarrow k_1(k_1')^-$ and $\alpha_2(k_2')^- : k_2(k_2')^- \Rightarrow h_2$. Therefore, the composite

$$(\alpha_1^-(k_1')^-) \star_1 (\beta_1 \star_0 ((k_1')^- \beta_2^- (k_2')^-)) \star_1 (\alpha_2(k_2')^-)$$

is a 2-cell of $\Gamma^\top$ with source h_1 and target h_2, proving that h_1 and h_2 are Γ-congruent. $\square$

7.1.18. Remark. The proofs of Theorems 7.1.4 and 7.1.7 are similar to the consistency proofs for 1-polygarphs. Indeed, if we forget the 2-dimensional coherence cells and look only at their 1-dimensional borders in (7.1.25), (7.1.6) and (7.1.8), we obtain precisely the diagrams used to prove the analogous 1-dimensional consistency results. This shows that the higher dimensional approach is consistent with the abstract case while providing several advantages. Notably, since the higher-dimensional cells may be considered as rewriting systems in their own right, and since the procedures described above work in any dimension, higher-dimensional rewriting provides a constructive method for calculating resolutions and cofibrant replacements of algebraic structures [65, 67].

Instead of using Proposition 7.1.14, we can use the notion of rewriting strategy to prove the above theorem. This has the added advantage that strategies provide a notion of normal forms for the higher dimensional rewriting system defined by coherence cells.

7.1.19. Sections and strategies. Let P be a 1-polygraph. Recall from Section 5.2.2 that we obtain a quotient set $q : P_0 \to \overline{P}_0$ by identifying 0-cells connected by zig-zag sequences. A*section* of P is a map $s : \overline{P}_0 \to P_0$ such that $q \circ s = id_{\overline{P}_0}$.

When P is a convergent ARS, we have existence and unicity of normal forms, so the map $x \mapsto \hat{x}$ is a section of P, which we call the *normal forms section*.

Given a section s of P, a *strategy* for P relative to s is a map

$$\begin{aligned} \sigma : P_0 &\longrightarrow P_1^* \\ x &\longmapsto \sigma_x \end{aligned}$$

which must satisfy the following for all $x \in P_0$:

i) $\sigma_x : x \to s(x)$,

ii) $\sigma_{s(x)} = 1_{s(x)}$.

When P is a convergent ARS, we call any strategy for P relative to the normal forms section a *normalisation strategy.*

7.1.20. Strategic confluence. We now define notions of Γ-confluence for a 1-polygraph P with respect to a (normalisation) strategy σ.

i) A branching (f, g) is *strategically* Γ-*confluent* provided that there exists a 2-cell α as in the following diagram:

$$\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x_1 \arrow[dr, "\sigma_{x_1}"'] \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & x_2 \arrow[dl, "\sigma_{x_2}"] \\
 & \hat{x} &
\end{tikzcd}$$

We say that P is *strategically* Γ-*confluent* when all of its branchings are.

ii) A zig-zag sequence $h : u \leftrightarrow v$ of P is *strategically* Γ-*confluent* provided that there exists a 2-cell $\alpha \in \Gamma^\top$ as in the following diagram:

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h"] \arrow[dr, "\sigma_u"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"', shift right=3] & & v \arrow[dl, "\sigma_v"] \\
 & \hat{u} = \hat{v} &
\end{tikzcd}$$

We say that P is *strategically* Γ-*Church-Rosser* if every zig-zag sequence of P is strategically Γ-confluent.

7.1.21. Squier's theorem with normalisation strategies. Here we prove Theorem 7.1.16 with a different technique, namely using normalisation strategies. This again proceeds in two steps: first we prove a version of Newman's lemma stating that a locally confluent, terminating ARS equipped with its family of generating confluences is such that any confluence may be paved toward a confluence in the strategy. The second step then becomes much simpler, being essentially the Church-Rosser argument.

7.1.22. Proposition (Strategic Newman's Lemma). *Let P be a locally confluent, terminating 1-polygraph and Γ a family of generating confluences. Then P is strategically Γ-confluent.*

Proof. The proof is essentially that of Theorem 7.1.7, but the Noetherian, induction hypothesis integrates the strategic confluence. ☐

7.1.23. Proposition (Strategic Church-Rosser). *Let P be a locally confluent, terminating 1-polygraph and Γ a family of generating confluences. Then P is strategically Γ-Church-Rosser.*

Proof. The proof follows the schema of Theorem 7.1.4. However, we provide details since the use of strategies changes the composite 2-cells slightly. Let h be a zig-zag sequence in P.

We reason by induction on the length of h. If h is an identity, the confluence (σ_x, σ_x) is paved by the identity 2-cell 1_{σ_x}. Suppose now that h is of length $n \geq 1$. We decompose $h = h_1 \star_0 h_2$ where, as in Theorem 7.1.4, h_1 is a zig-zag of length $(n-1)$ and h_2 is a rewriting step or the inverse of a rewriting step. We again treat these two cases separately. For the first, we pave the diagram as illustrated below:

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h_1"] \arrow[dr, "\sigma_u"'] & \overset{\alpha}{\Longrightarrow} & w \arrow[rr, "h_2"] \arrow[dl, "\sigma_w"] & \overset{\beta}{\Longrightarrow} & v \arrow[dl, "\sigma_v"] \\
& \hat{u} = \hat{w} \arrow[rr, equal] & & \hat{w} = \hat{v} &
\end{tikzcd} \tag{7.1.24}$$

where α is obtained by the induction hypothesis and β is obtained by Proposition P:StratNewmanARS. The composite $\alpha \star_1 (h_1 \star_0 \beta)$ makes the zig-zag h Γ-confluent.

In the second case, we obtain a diagram of the form

$$\begin{tikzcd}
u \arrow[rr, leftrightarrow, "h_1"] \arrow[dr, "\sigma_u"'] & \overset{\alpha}{\Longrightarrow} & w \arrow[rr, "h_2^-"] \arrow[dl, "\sigma_w"] & \overset{\beta}{\Longrightarrow} & v \arrow[dl, "\sigma_v"] \\
& \hat{u} = \hat{w} \arrow[rr, equal] & & \hat{w} = \hat{v} &
\end{tikzcd} \tag{7.1.25}$$

The 2-cell α is again inherited from the induction hypothesis. For the 2-cell β, note that we obtain a 2-cell $\gamma : h_2\sigma_w \Rightarrow \sigma_v$ by applying Proposition 7.1.22 to the branching (h_2, σ_v). We obtain β by whiskering:

$$\beta = h_2^- \gamma : \sigma_w \Longrightarrow h_2^- \sigma_v.$$

Then β is a 2-cell whose 1-source is that of α, and whose 1-target is $h_2^- \sigma_v$. The composite 2-cell $\alpha \star_1 (h_1 \beta)$ concludes the proof.

□

Now we can provide an alternate proof of Theorem 7.1.16.

Proof. By Proposition 7.1.23, we know that P is strategically Γ-Church-Rosser. We thereby obtain the 2-cells α_1 and α_2 illustrated below:

$$\begin{tikzcd}
& v \arrow[drr, bend left, "1_v"] \arrow[dr, "\sigma_v"] & \| & \\
u \arrow[ur, leftrightarrow, bend left, "h_1"] \arrow[dr, leftrightarrow, bend right, "h_2"'] \arrow[rr, "\sigma_u"] & \overset{\alpha_1 \Uparrow}{\underset{\Downarrow \alpha_2}{}} & \hat{u} = \hat{v} \arrow[r, "\sigma_v^-"] & v \\
& v \arrow[ur, "\sigma_v"'] \arrow[urr, bend right, "1_v"'] & \| &
\end{tikzcd} \tag{7.1.26}$$

The composite 2-cell $(\alpha_1 \sigma_v^-)^- \star_1 (\alpha_2 \sigma_v^-)$ provides the conclusion. □

7.2. COHERENCE FOR STRING REWRITING SYSTEMS

Here we provide a brief description of the coherent critical branching lemma in the language of 3-polygraphs.

We define the notion of a *homotopy basis* of a 2-polygraph P in a similar fashion to the one-dimensional case. This consists of a cellular extension P_3 of $P_1^*(P_2)$ such that the quotient 2-category $P_1^*(P_2)/P_3$ is acyclic, *i.e.* every 2-sphere of $P_1^*(P_2)/P_3$ is of the form $(\overline{\alpha}, \overline{\alpha})$ where α is a 2-cell of $P_1^*(P_2)$. For example, the set of all 2-spheres of $P_1^*(P_2)$ is (trivially) a homotopy basis of P. A *coherent presentation* of a category $\mathcal{C}$ is a tuple (P_0, P_1, P_2, P_3) where P_3 is a homotopy basis of $P = (P_0, P_1, P_2)$, a 2-polygraph presenting $\mathcal{C}$.

For a convergent 2-polygraph P, a *family of generating critical confluences* of P is a cellular extension of the free $(2,1)$-category $P_1^*(P_2)$ containing one 3-cell for every critical branching (α, β)

$$\begin{tikzcd}
 & u \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "\beta"] & \\
x_1 \arrow[rr, phantom, "\overset{\alpha}{\Rrightarrow}"] \arrow[dr, Rightarrow, "\alpha'"'] & & x_2 \arrow[dl, Rightarrow, "\beta'"] \\
 & \hat{x} &
\end{tikzcd}$$

Note that this family is not unique, since we arbitrarily choose the source and target, *i.e.* direction, of each 3-cell. Similarly to the distinction between consistency theorems for ARS and SRS, we can link the critical branchings of P to the existence of a homotopy basis thereof:

7.2.1. Theorem. *For a convergent polygraph P, every family of generating critical confluences is a homotopy basis of $P^\top$.*

7.3. COHERENCE IN HIGHER DIMENSIONAL REWRITING SYSTEMS

Here we generalise the theorems we saw in the first section of this chapter. The coherent abstract rewriting proofs that we described for 1-polygraphs may also be interpreted in higher dimensional rewriting systems, that is, n-polygraphs.

7.3.1. Γ-confluence. Recall from [64] that, given an n-polygraph P and a cellular extension Γ of $P^\top$, we say that P is Γ-*confluent* (resp. *locally* Γ-*confluent*) if for any branching (resp. local branching) (f, g) of P there exists a confluence (f', g') completing the branching, and an $(n+1)$-cell α in the free $(n+1, n-1)$-category $P^\top(\Gamma)$ of the form $\alpha : f \star_{n-1} f' \to g \star_{n-1} g'$. Similarly, we say that P is Γ-*Church-Rosser* if for any n-cell f of $P_n^\top$ there exists a confluence (f', g') rendering f confluent, and an $(n+1)$-cell α in the free $(n+1, n-1)$-category $P_n^\top(\Gamma)$ of the form $\alpha : f \star_{n-1} f' \to g'$.

7.3.2. Family of generating confluences. Let P be a locally confluent n-polygraph. As in the case of 1-polygraphs, a *family of generating confluences* for P is a cellular extension Γ consisting of a, $(n+1)$-cell $\alpha_{f,g}$ for every local branching (f,g) and the choice of a confluence (f',g') as in the following diagram:

$$\begin{tikzcd}
& x \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x_1 \arrow[rr, Rightarrow, "\alpha_{f,g}"] \arrow[dr, "f'"'] & & x_2 \arrow[dl, "g'"] \\
& x' &
\end{tikzcd}$$

7.3.3. Coherence for n-polygraphs. Let P be a $(2,0)$-polygraph. We say that Γ is a *homotopy basis* if for all parallel zig-zag sequences f and g in P, *i.e.* those with common source and target, there exists a $(n+1)$-cell α in $P^\top(\Gamma)$ such that $s_n(\alpha)=f$ and $t_n(\alpha)=g$. Symbolically and diagrammatically stated:

$$\forall f,g \in P_1^\top, \begin{cases} s_{n-1}(f)=s_{n-1}(g) \\ t_{n-1}(f)=t_{n-1}(g) \end{cases} \quad \exists \alpha \in P^\top(\Gamma)_{n+1}, \alpha : f \Rrightarrow g. \quad \begin{tikzcd} \cdot \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \cdot \end{tikzcd}$$

A homotopy basis is also called a *coherent extension* of P.

7.3.4. Underlying $(2,0)$-polygraph. The coherence theorem for n-polygraphs [64] can be proved using the results from the previous section, applied to a $(2,0)$-polygraph derived from the n-polygraph in question. Throughout this section, we fix a n-polygraph P and a cellular extension Γ of $P^\top = P_0[P_1]\ldots[P_{n-1}](P_n)$.

Recall from Section 5.4.12 that the underlying rewriting polygraph P^c associated to P is given by (P_{n-1}^*, P_n^c). We denote by $(P^c)^\top$ the 1-groupoid generated by this 1-polygraph. The cellular extension Γ of $P^\top$ can equally be seen as a cellular extension of $(P^c)^\top$, as a result of Remark 5.4.9. Indeed, recall that the 1-cells of $(P^c)^\top$ are in bijective correspondence with the n-cells of $P^\top$. Given an element $\alpha : f \to g$ of Γ, we may interpret it as a 2-cell with source (resp. target) the 1-cell of $(P^c)^\top$ corresponding to f (resp. g). We denote this cellular extension by Γ_2, to highlight the shift in dimension.

Due to this correspondence, the notions of (local) Γ-confluence and Γ-Church-Rosser for P coincide with the notions of (local) Γ_2-confluence for P^c.

7.3.5. Proposition. *Let P be an n-polygraph. Then P is (locally) Γ-confluent (resp. Γ-Church-Rosser) if, and only if P^c is (locally) Γ_2-confluent (resp. Γ_2-Church-Rosser)*

Notions of sections and strategies are also given by the underlying rewriting polygraph associated to P. We thereby also obtain notions of strategic Γ-confluence and strategic Γ-Church-Rosser in the n-dimensional context.

This gives us the following (abstract) coherence theorem for n-polygraphs, where we can use both the strategic and non-strategic proof.

7.3.6. Theorem (Coherence theorem for ARS). *Let P be a locally confluent, terminating n-polygraph and Γ a family of generating confluences. Then Γ is a homotopy basis for P.*

CHAPTER 8.
ALGEBRAIC ABSTRACT COHERENCE

The goal of this chapter is to formulate and prove the coherence theorem for ARS, Theorem 7.1.16, in an algebraic context generalising the abstract rewriting paradigm provided by Kleene algebras. For this, we introduce the notion of 2-Kleene algebra, and equip it with modalities and other algebraic structure. For now, we stay in the context of two dimensional structures, since as was pointed out in Section 7.3.4, the coherence theorem (without the critical branching lemma) only requires three types of information, namely the objects and reduction rules, which are zero and one dimensional respectively, and coherence cells, which are then two dimensional. In Chapter 9 we treat the case of n-Kleene algebra, here we provide the minimal axioms and results which allow us to state and prove the coherence theorem.

We start in Section 8.1 by introducing an alternative paving mechanism for coherence proofs in the context of n-polygraphs. While the methods described in Chapter 7 use coherence cells to relate rewriting paths, we describe an alternative description in which coherence cells relate zig-zag sequences. This becomes important when we introduce globular 2-Kleene algebras and their coherent rewriting properties in Sections 8.2 and 8.3, where this "vertical" approach is more natural. In the former, we also provide models of globular 2-Kleene algebras as liftings of polygraphs to the power-set.

In Section 8.4, we capture the notion of strategy, recalled in the polygraphic setting in Section 7.1.19, and prove lemmas relating these to rewriting properties. Finally, Kleene algebraic versions of the coherent normalising Newman's lemma is formulated and proved, leading to Theorem 8.5.2, the Kleene algebraic coherence theorem in the context of abstract rewriting.

Results and definitions in this chapter are along the lines of [16] and constitute personal contributions. As indicated above, a more extensive treatment of higher Kleene algebras, as introduced in [17] can be found in Chapter 9; here we provide a minimal description in order to simplify the road towards Theorem 8.5.2.

8.1. A VERTICAL APPROACH TO COHERENCE

In Section 6.2, the rewriting properties of Kleene algebra were described. Recall that in that context, the universal and existential quantification on branchings and confluences, respectively, was encoded by inequalities which we called semi-commutations. This means that the flow of proofs move from branchings to confluences, *i.e. vertically.* On the other hand, in Section 7.1.3, we saw that in the polygraphic context, the usual direction of higher cells in coherence proofs are *horizontal.* In order to describe how coherent rewriting techniques may be expressed in Kleene algebraic structures, we explain how this vertical approach to coherence is instantiated in the polygraphic context.

Although the following sections focus on 2-Kleene algebras, we explain this approach in the context of n-polygraphs in order to relate it to higher Kleene algebras in Chapter 9.

8.1.1. Coherent confluence. Let P be an n-polygraph and Γ a cellular extension of $P_n^\top$.

The cellular extension Γ is a *confluence filler* of a branching (f, g) of P if there exist rewriting paths f', g' of P as in (8.1.2), and two $(n+1)$-cells α, α' in the free $(n+1)$-category $P_n^\top[\Gamma]$ of the form $\alpha : f^- \star_{n-1} g \to f' \star_{n-1} (g')^-$ and $\alpha' : g^- \star_{n-1} f \to g' \star_{n-1} (f')^-$:

$$\begin{tikzcd}
& u \arrow[dl, "f"'] \arrow[dr, "g"] & & & u \arrow[dr, "g"] & & & u \arrow[dl, "f"'] & \\
u_1 \arrow[dr, "f'"'] & & v_1 \arrow[dl, "g'"] & u_1 \arrow[ur, "f^-"] \arrow[dr, "f'"'] & \Downarrow\alpha & v_1 & u_1 & \Downarrow\alpha' & v_1 \arrow[ul, "g^-"'] \arrow[dl, "g'"] \\
& u' & & & u' \arrow[ur, "(g')^-"'] & & & u' \arrow[ul, "(f')^-"] &
\end{tikzcd} \tag{8.1.2}$$

In this case, α and α' are n-compositions of $(n+1)$-cells of Γ^c as recalled in Remark 5.4.9. We say that the cellular extension Γ is a *confluence filler* for the polygraph P if Γ is a confluence filler for each of its branchings.

More generally, the cellular extension Γ is a *confluence filler* of an n-cell f in $P_n^\top$ if there exist n-cells f' and g' in P_n^* and an $(n+1)$-cell α in the free $(n+1)$-category $P_n^\top[\Gamma]$ of the form $\alpha : f \to f' \star_{n-1} g'^-$:

$$\begin{tikzcd}
u \arrow[rr, "f"] \arrow[dr, "f'"'] & & v \arrow[dl, "g'"] & u \arrow[rr, "f"] \arrow[dr, "f'"'] & \Downarrow\alpha & v \\
& u' & & & u' \arrow[ur, "(g')^-"'] &
\end{tikzcd} \tag{8.1.3}$$

The cellular extension Γ is a *Church-Rosser filler* for an n-polygraph P when it is a confluence filler of every n-cell in $P_n^\top$.

8.1.4. Theorem (Church-Rosser coherent filler lemma). *Let P be an n-polygraph. A cellular extension Γ of $P_n^\top$ is a confluence filler for P if, and only if, Γ is a Church-Rosser filler for P.*

Proof. First suppose that Γ is a Church-Rosser filler for P. Given a branching (f, g), we have that $f^- \star_{n-1} g$ and $g^- \star_{n-1} f$ are elements of $P_n^\top$ and thus Γ is a confluence filler for these n-cells. This gives us the cells α and α' as in (8.1.2), and so Γ is a confluence filler for P.

Conversely, suppose that Γ is a confluence filler for P, and let $f \in P_n^\top$ be an n-cell. We prove by induction on the length of f that Γ is a Church-Rosser filler for P. For f of length 0 or 1, we clearly have that f is Γ-confluent, since it suffices to take an identity $(n+1)$-cell. Suppose that every n-cell of length $k \geq 2$ is Γ-confluent and that f is of length $k+1$. Then $f = f_1 \star_{n-1} f_2$ with $f_1 : u \to u_1$ in $P_n^\top$ of length k and f_2 is of length 1 in P_n^* either of the form $v \to u_1$ or $u_1 \to v$. By the induction hypothesis there exist rewriting paths h and k and an $(n+1)$-cell α such that $\alpha : f \Rightarrow hk^-$. If $f_2 : u_1 \to v$, there exist rewriting paths k' and f'' and an $(n+1)$-cell β as depicted in diagram (8.1.5) since Γ is a confluence filler for P. Thus $(\alpha f_2) \star_n (h\beta)$ is a confluence filler for f.

$$\begin{tikzcd}
u \arrow[rr, tail, "f_1"] \arrow[dr, "h"'] & \arrow[d, phantom, "\Downarrow\alpha"] & u_1 \arrow[rr, "f_2"] & \arrow[d, phantom, "\Downarrow\beta"] & v \\
& u' \arrow[ur, "k^-"] \arrow[rr, "k'"'] & & u'' \arrow[ur, "f''^-"'] &
\end{tikzcd} \tag{8.1.5}$$

If $f_2 : v \to u_1$, the $(n+1)$-cell $\alpha f_2^- \star_n h1_{k^-(f_2)^-} = \alpha f_2^-$ is a confluence filler for f.

$$\begin{tikzcd}
u \arrow[rr, tail, "f_1"] \arrow[dr, "h"'] & \arrow[d, phantom, "\Downarrow\alpha"] & u_1 \arrow[rr, "(f_2)^-"] & \arrow[d, phantom, "\Downarrow 1_{k^-(f_2)^-}"] & v \\
& u' \arrow[ur, "k^-"] \arrow[rr, "(f_2)^-"'] & & u'' \arrow[ur, "k^-"'] &
\end{tikzcd} \tag{8.1.6}$$

□

8.1.7. Theorem (Coherent Newman filler lemma). *Let P be a terminating n-polygraph, and Γ a cellular extension of $P_n^\top$. Then Γ is a local confluence filler, if, and only if, Γ is a confluence filler for P.*

Proof. Firstly, observe that if Γ is a confluence filler for P, then it is also a local confluence filler for P since local branchings are also branchings.

Now suppose that Γ is a local confluence filler for P. We prove by Noetherian induction that, for every $(n-1)$-cell u of P_n^*, Γ is a confluence filler for every branching of P with source u. For the base case, if u is irreducible for P, then $(1_u, 1_u)$ is the only branching with source u, and it is Γ-confluent, taking the $(n+1)$-cell 1_{1_u}.

Suppose now the induction hypothesis, namely that u is a reducible $(n-1)$-cell of P_n^* and that Γ is a confluence filler for every branching with source an $(n-1)$-cell u' such that u rewrites to u'. Let (f, g) be a branching of P with source u. If one of f or g is an identity, say f, then Γ is a confluence filler for (f, g) by considering the $(n+1)$-cells 1_g and 1_{g^-}. We may now suppose that the n-cells f and g are not identities, thus we write

$f = f_1 \star_{n-1} f_2$ and $g = g_1 \star_{n-1} g_2$, where g_1, f_1 are rewriting steps and g_2, f_2 are n-cells of P_n^*. Since Γ is a local confluence filler for P, there exist n-cells f_1', g_1' in P_n^*, and an $(n+1)$-cell α in $P_n^*[\Gamma]$ as in the diagram (8.1.8). We apply the induction hypothesis to the branching (f_2, f_1'), which yields n-cells f_2', h in P_n^* and an $(n+1)$-cell β in $P_n^*[\Gamma]$ as in the diagram (8.1.8). Finally, we apply the induction hypothesis again to the branching $(g_1' \star_{n-1} h, g_2)$ yielding n-cells k and g_2' and an $(n+1)$-cell γ in $P_n^*[\Gamma]$ as in (8.1.8).

$$\begin{tikzcd}
 & & u \arrow[dr, "g_1"] & & \\
 & u_1 \arrow[ur, "f_1^-"] \arrow[dr, "f_1'"'] & \Downarrow \alpha & v_1 \arrow[dr, "g_2"] & \\
u_2 \arrow[ur, "f_2^-"] \arrow[dr, "f_2'"'] & \Downarrow \beta & u' \arrow[ur, "(g_1')^-"'] & & v_2 \\
 & u_2' \arrow[ur, "h^-"] \arrow[dr, "k"'] & & \Downarrow \gamma & \\
 & & u'' \arrow[uurr, "(g_2')^-"'] & &
\end{tikzcd} \tag{8.1.8}$$

The n-composition

$$\delta = (((f_2^- \star_{n-1} \alpha) \star_n (\beta \star_{n-1} (g_1')^-)) \star_{n-1} g_2) \star_n (f_2' \star_{n-1} \gamma) \tag{8.1.9}$$

is an $(n+1)$-cell in $P_n^*[\Gamma]$ with source $f^- \star_{n-1} g$ and target $f_2' \star_{n-1} k \star_{n-1} (g_2')^-$. We can similarly find an $(n+1)$-cell δ' with source $g^- \star_{n-1} f$ and with target a confluence. Γ is thus a confluence filler for P, which proves the result. □

8.1.10. Remark. We have defined two approaches to coherence properties of an n-polygraph P with respect to a cellular extension Γ:

i) A *"vertical" approach* in which the coherence cells, *i.e.* the $(n+1)$-cells generated by Γ, have a branching as n-source and a confluence as n-target. This necessitates having inverses of n-cells, that is Γ is a cellular extension of $P_n^\top$. For example, in the proofs of Theorems 8.1.4 and 8.1.7, we do not need inverses of $(n+1)$-cells.

ii) A *"horizontal" approach* in which coherence cells have rewriting paths for both source and target, and we do not need inverses of n-cells, *i.e.* we consider cellular extensions of P_n^*, only inverses of $(n+1)$-cells in order to prove Theorems 7.1.4 and 7.1.7, which yield the higher dimensional analogues as described in Section 7.3.4.

These differences can be summed up by saying that, in the first approach, the proofs take place in $P_n^\top[\Gamma]$, whereas, in the second one, the proofs take place in $P_n^*(\Gamma)$.

Furthermore, it is worth noting that, in the first approach, we specify two filler cells α and α' as depicted in diagram 8.1.2 for each branching (f, g). This is due to the fact that branchings are unordered pairs, so we must account for both cases. This equally constitutes the reason we require inverses of $(n+1)$-cells in the second approach.

In the rest of this document, we will exclusively consider the vertical approach to paving diagrams with higher dimensional cells. The motivation of this choice lies in the fact that with Kleene algebras, we pave diagrams from a *relational* rather than a *polygraphic* point of view. We thus follow the direction of the n-cells in branchings and confluences, *i.e.* vertically. This is a consequence of the quantification on branchings and confluences: we quantify *universally* over branchings and *existentially* over confluences. In the polygraphic approach, this quantification is hidden by specifying the *choice* of $(n+1)$-cells filling confluence diagrams.

8.2. GLOBULAR 2-KLEENE ALGEBRAS

Here we introduce globular 2-Kleene algebras, a natural extension of Kleene algebra, in order to formulate and prove coherence theorems for abstract rewriting systems in a point-free algebraic setting. These constitute a special case of globular n-Kleene algebras, introduced in [17], see Chapter 9.

8.2.1. 2-dioids. A 2-dioid is a structure

$$(S, +, 0, \odot_0, 1_0, \odot_1, 1_1)$$

such that for each $i \in \{0,1\}$, the tuple $(S, +, 0, \odot_i, 1_i)$ is a dioid and such that the following axioms hold:

- The *lax interchange law*: for all $A, A', B, B' \in K$,

$$(A \odot_1 A') \odot_0 (B \odot_1 B') \leq (A \odot_0 B) \odot_1 (A' \odot_0 B'). \tag{8.2.1}$$

- The 1-unit is an idempotent for 0-multiplication, that is

$$1_1 \odot_0 1_1 = 1_1. \tag{8.2.2}$$

For $i \in \{0,1\}$, we refer to $\odot_i$ and 1_i as the i-multiplication and the i-unit, respectively. In contrast to the equational case, the lax interchange law does not incur an Eckmann-Hilton collapse. Note that these axioms correspond to those of concurrent semirings [75], except that the equality $1_0 = 1_1$ is normally assumed in the concurrent case.

The above axioms provide an underlying algebraic structure for reasoning about coherence in the context of abstract rewriting. The lax interchange law corresponds to the lifting of the equational interchange law for compositions in a 2-category, while the idempotence of the 1-unit expresses the fact that the set of 1-cells in a 2-category are closed under 0-composition.

In what follows, we fix a 2-semiring $(S, +, 0, \odot_i, 1_i)_{i=0,1}$.

8.2.2. Domain 2-semirings. We now augment S with multiple domain maps. We say that S is a *domain 2-semiring* if it is equipped with maps

$$d_0 : S \to S \qquad \text{and} \qquad d_1 : S \to S$$

called 0- *and* 1-*domain*, respectively. These must satisfy that for each $i \in \{0, 1\}$, $(S, +, 0, \odot_i, 1_i, d_i)$ is a domain semiring, and the *absorption axiom*

$$d_1 \circ d_0 = d_0. \tag{8.2.3}$$

For $i \in \{0, 1\}$, the set $S_{d_i} = d_i(S)$ will be called the *i-domain algebra*, and will be denoted by S_i. The absorption axiom (8.2.3) implies that $S_0 \subseteq S_1$, *i.e.* we have a hierarchization of domain algebras. This inclusion is akin to viewing, for example 0-cells in a category as identity 1-cells. For this reason, elements of S_i will be referred to as *i-dimensional.*

Furthermore, to distinguish elements of distinct dimensions, we henceforth denote elements of S_0 by $p, q, r, \ldots$, elements of S_1 by $\phi, \psi, \xi, \ldots$, and other elements of S by $A, B, C, \ldots$ This notation simplifies the reading of proofs when elements of different dimensions are interacting.

8.2.3. Boolean domain 2-semirings. We may also augment the domain algebras with a Boolean structure. For $p \in \{0, 1\}$ we say that S is *p-Boolean* if it is augmented with $(p + 1)$ maps

$$(ad_i : S \to S)_{0 \leq i \leq p}$$

such that for all $0 \leq i \leq p$, the following conditions are satisfied:

i) $(S, +, 0, \odot_i, 1_i, ad_i)$ is a Boolean domain semiring,

ii) $d_i = ad_i^2$.

The map ad_i is called i-antidomain because, as recalled in Section , the restriction of ad_i to S_i provides the latter with the structure of a Boolean algebra. For this reason, we denote this restriction by $\neg_i$. By definition, a 0-Boolean domain 1-semiring is a Boolean domain semiring, and by convention we say that a 0-*Boolean domain* 0-*semiring* is a Boolean algebra.

8.2.4. Modal 2-semirings. As in the case of 1-semirings, we also define a notion of codomains. Denote by S^{op} the 2-semiring in which the order of multiplication has been reversed. We say that S is a *(p-Boolean) codomain* 2-*semiring* if S^{op} is a (p-Boolean) domain 2-semiring. We will denote the codomain maps by r_i. The *codomain algebras* are defined analogously to those given by domain, and will temporarily be denoted by S_{r_i}.

We say that S is *modal* if it is a domain and codomain 2-semiring and the following compatibility axioms hold for all $i \in \{0, 1\}$:

$$d_i \circ r_i = r_i \quad \text{and} \quad r_i \circ d_i = d_i. \tag{8.2.5}$$

These axioms ensure that the domain and codomain algebras coincide, *i.e.* that $S_{d_i} = S_{r_i}$. For this reason, we will maintain the moniker i-domain algebra and the notation S_i.

We say that S is a p-Boolean modal semiring if it is p-Boolean with respect to both domain and codomain. In this case, the compatibility axiom (8.2.5) is not necessary, as was pointed out in Section 6.1.5 .

8.2.6. Modal operators. The *i-diamond operators* of a modal 2-Kleene algebra K are defined via the (co-)domain operators in each dimension. For $i \in \{0,1\}$, $A \in K$ and $\phi \in K_i$,

$$|A\rangle_i(\phi) = d_i(A \odot_i \phi), \quad \text{and} \quad \langle A|_i(\phi) = r_i(\phi \odot_i A).$$

These modal operators have all of the properties recalled in Section 6.1 with respect to i-operations and elements of K_i.

In the case of a p-Boolean modal Kleene algebra, we may also define *i-box operators* for all $i \leq p$. For $A \in K$ and $\phi \in K_i$,

$$|A]_i(\phi) := \neg_i|A\rangle_i(\neg_i\phi), \quad \text{and} \quad [A|_i(\phi) = \neg_i\langle A|_i(\neg_i\phi).$$

8.2.7. Converses. Throughout this chapter, we will consider modal 2-Kleene algebras with *0-converses*, *i.e.* equipped with an operation $\overline{(-)} : K_1 \to K_1$ such that

$$(K_1, +, 0, \odot_0, 1_0, (-)^{*_0}, \overline{(-)})$$

is a MKA with contracting converse, as defined in Section 6.1.17. For a more general notion of converse in higher-dimensional Kleene algebra, we refer the reader to Section 9.5.1.

8.2.8. Globularity. A modal 2-Kleene algebra K is *globular* if the following *globular relations* hold for all $A, B \in K$:

$$\begin{aligned} d_0 \circ d_1 = d_0 \ \text{ and } \ d_0 \circ r_1 = d_0, & \qquad d_1(A \odot_0 B) = d_1(A) \odot_0 d_1(B), \\ r_0 \circ d_1 = r_0, \ \text{ and } \ r_0 \circ r_1 = r_0, & \qquad r_1(A \odot_0 B) = r_1(A) \odot_0 r_1(B). \end{aligned}$$

As a consequence of the rightmost axioms, K_1 is a MKA with respect to 0-operations. An element A of K will be represented graphically by the below diagram with respect to its 0- and 1-domains and codomains.

$$\begin{tikzcd} d_0(A) \arrow[r, bend left, "d_1(A)"] \arrow[r, bend right, "r_1(A)"'] \arrow[r, phantom, "\Downarrow A"] & r_0(A) \end{tikzcd}$$

8.2.9. Modal 2-Kleene algebras. The 1-star $(-)^{*_1}$ is a *lax morphism* with respect to 0-multiplication of 1-dimensional elements on the right (resp. left), *i.e.* for all $A \in K$ and $\phi \in K_1$,

$$\phi \odot_0 A^{*_1} \leq (\phi \odot_0 A)^{*_1}, \qquad (\text{resp. } A^{*_1} \odot_0 \phi \leq (A \odot_0 \phi)^{*_1}).$$

In order to distinguish elements of distinct dimensions, we denote elements of K_0 by $p, q, r, \ldots$, elements of K_1 by $\phi, \psi, \xi, \ldots$, and elements of K of non-determinate dimension by $A, B, C, \ldots$

8.2.10. Polygraphic model. Here we provide a minimal description of the polygraphic model of 2-Kleene algebras in order to provide intuition. A full treatment can be found in Section 9.1.19.

Let (Φ, X) be a $(2,0)$-polygraph. We define $K(\Phi, X)$, the *full* 2-*path algebra over* (Φ, X) as follows. Let $X_2^\top$ denote the set of 2-cells in $X^\top$. The carrier set of $K(\Phi, X)$ is the power set $\mathcal{P}(X_2^\top)$, whose elements, denoted by $A, B, C \ldots$ are sets of 2-cells, which in turn are denoted by $\alpha, \beta, \gamma \ldots$ Recall that for each 1-cell x of $X^\top$, there exists a unique 2-cell 1_x, its identity 2-cell, and similarly, for each 0-cell a there exists a unique 2-cell 1_{1_a}, the identity 2-cell on its identity 1-cell. For $i \in \{0,1\}$, the i-composition, i-source and i-target maps are thereby defined for cells of any dimension.

For $i \in \{0,1\}$, the multiplication $\odot_i$ on $K(\Phi, X)$ is the lifting of the composition operations of $X^\top$ to the power-set, *i.e.* for any $A, B \in K(\Phi, X)$,

$$A \odot_i B := \{\alpha \star_i \beta \mid \alpha \in A \wedge \beta \in B \wedge t_i(\alpha) = s_i(\beta)\}.$$

The units are the sets $\mathbb{1}_0 = \{1_{1_a} \mid a \in \Phi_0\}$, and $\mathbb{1}_1 = \{1_x \mid x \in \Phi_1^\top\}$. The addition in $K(\Phi, X)$ is given by set union; the ordering is therefore given by set inclusion. The domain and codomain maps are defined by

$$\begin{aligned} d_0(A) &:= \{1_{1_{s_0(\alpha)}} \mid \alpha \in A\}, & r_0(A) &:= \{1_{1_{t_0(\alpha)}} \mid \alpha \in A\}, \\ d_1(A) &:= \{1_{s_1(\alpha)} \mid \alpha \in A\}, \quad \text{and} & r_1(A) &:= \{1_{t_1(\alpha)} \mid \alpha \in A\}, \end{aligned}$$

and are thus given by lifting the source and target maps of $X^\top$ to the power set. The i-antidomain and i-anticodomain maps are then given by complementation with respect to the set of i-cells. The i-star is given by $A^{*_i} = \bigcup_{k \in \mathbb{N}} A^{k_i}$, where in the above, $A^{0_i} := \mathbb{1}_i$ and $A^{k_i} := A \odot_i A^{(k-1)_i}$. For $\psi \in K(\Phi, X)_1$, the converse is given by $\overline{\psi} := \{1_{x^-} \mid 1_x \in \psi\}$.

8.2.11. Proposition ([17]). *Let (Φ, X) be a $(2,0)$-polygraph. Then, $K(\Phi, X)$ is a globular Boolean modal 2-Kleene algebra.*

Proof. See the proof of Proposition 8.2.11, of which this is a special case. □

8.3. Coherent rewriting and modal 2-Kleene algebras

Here, we recall a minimal account of coherent rewriting properties in globular 2-Kleene algebras [17]. A more complete treatment of coherent rewriting mechanisms in higher Kleene algebras can be found in Sections 9.2 and 9.3.

8.3.1. Fillers. We fix K a globular 2-Kleene algebra. Given $A \in K$ and $\phi, \phi' \in K_1$, $|A\rangle_1(\phi) \geq \phi'$ is equivalent to $d_1(A \odot_1 \phi) \geq \phi'$ by definition. In terms of quantification over collections of cells, this means that *for every* u in ϕ', *there exist* v in ϕ and α in A such that the 1-source (resp. 1-target) of α is u (resp. v). This observation motivates the following definitions from [17]. For ϕ, ψ in K_1, an element A in K is

- a *local confluence filler* for (ϕ, ψ) if $|A\rangle_1(\psi^{*_0} \odot_0 \phi^{*_0}) \geq \phi \odot_0 \psi$,
- is a *confluence filler* for (ϕ, ψ) if $|A\rangle_1(\psi^{*_0} \odot_0 \phi^{*_0}) \geq \phi^{*_0} \odot_0 \psi^{*_0}$,
- and is a *Church-Rosser filler* for (ϕ, ψ) if $|A\rangle_1(\psi^{*_0} \odot_0 \phi^{*_0}) \geq (\psi + \phi)^{*_0}$.

8.3.2. Whiskers and completion. The *right* (resp. *left*) *whiskering* of an element $A \in K$ by $\phi \in K_1$ is the element $A \odot_0 \phi$ (resp. $\phi \odot_0 A$). Recall from [17] that whiskering commutes with 1-diamonds, that is, for all $A \in K$ and $\phi, \psi, \phi', \psi', \gamma \in K_1$ such that $\phi' \leq \phi$, $\psi' \leq \psi$, and $d_1(A) \leq \gamma$, we have:

$$\phi' \odot_0 |A\rangle_1(\gamma) \odot_0 \psi' = |\phi' \odot_0 A \odot_0 \psi'\rangle_1(\phi \odot_0 \gamma \odot_0 \psi). \tag{8.3.1}$$

Fix a (local) confluence filler A of a pair (ϕ, ψ) of elements in K_1. The *total whiskering of* A, denoted by $\hat{A}$, is the following element of K:

$$\hat{A} := (\phi + \psi)^{*_0} \odot_0 A \odot_0 (\phi + \psi)^{*_0}. \tag{8.3.2}$$

8.3.3. Completion. The 1-star of $\hat{A}$ is called the *completion* of A. Note that this element *absorbs whiskers*, that is, for every $\xi \leq (\phi + \psi)^{*_0}$,

$$\xi \odot_0 \hat{A}^{*_1} \leq \hat{A}^{*_1} \qquad \text{and} \qquad \hat{A}^{*_1} \odot_0 \xi \leq \hat{A}^{*_1}. \tag{8.3.3}$$

8.4. Formalisation of normalisation strategies

In this section, we formalise the notion of normalisation strategy, introduced in [67]. We first define notions of section, skeleton and strategy in one-dimensional Kleene algebras and show properties thereof [16]. In what follows, we consider a Boolean MKA K with converse and an element $x \in K$.

8.4.1. Sections, skeleta and strategies.

- **i)** The *equivalence* generated by x is the element $x^\top := (x+\overline{x})^*$. For $p \in K_d$, the *x-saturation* of p is the element $|x^\top\rangle(p) \in K_d$.
- **ii)** A *covering set* for x is an element $q \in K_d$ such that $|x^\top\rangle(q) \geq 1$, *i.e.* whose x-saturation is total. A *section* of x is a minimal covering set.
- **iii)** A *wide sub* of x is an element $w \leq x$ such that $|w\rangle = |x\rangle$ and $\langle w| = \langle x|$. A *skeleton* of x is a minimal wide sub.
- **iv)** Given a section s_0 of x, a *strategy for x relative to s_0* is a skeleton σ of $x^\top s_0$ such that $s_0\sigma \leq s_0$.

8.4.2. Remark. Note that when (Φ, X) is a $(2,0)$-polygraph, we describe Φ in $K(\Phi, X)$ as the element $\phi := \{1_x \,|\, x \in \Phi_1\} \cup \{1_{1_a} \,|\, a \in \Phi_0\}$. In $K(\Phi, X)_1$, which we recall is a Boolean MKA for 0-operations, the equivalence generated by ϕ corresponds to the 1-groupoid $\Phi^\top$, and a section corresponds to a choice of a representative 0-cell for each connected component in $\Phi^\top$. A wide sub of ϕ is a subset ψ such that for any 1-cell $x : a \to b \in \Phi_1$, there exists some parallel 1-cell $x' : a \to b \in \Phi_1$ such that $1_{x'} \in \psi$. A skeleton of ϕ therefore corresponds to the choice of a single 1-cell amongst the sets of parallel 1-cells in Φ; it is thus not unique and does not coincide with ϕ in general. When Φ is convergent and $\{\sigma_a\}_{a\in\phi_0}$ is a strategy in the sense of Section 8.4, then $\sigma = \{1_{\sigma_a} | a \in \phi_0\}$ is a strategy for ϕ in $K(\phi, X)$ with respect to $\mathrm{NF}(\phi)$. This result is proved for any convergent element of a MKA in Proposition 8.4.5.

By definition, a strategy σ satisfies $d(\sigma) = d(x^\top s_0) = 1$, and $r(\sigma) = r(x^\top s_0) = s_0$. The following lemma states that a strategy contains the associated section:

8.4.3. Lemma ([16]). *Given a section s_0 of x and a strategy σ for x relative to s_0, we have $s_0\sigma = s_0$ and $s_0 \leq \sigma$.*

Proof. By hypothesis we have $s_0\sigma \leq s_0$. Showing that $s_0\sigma$ is a covering set allows us to deduce by minimality of s_0 that $s_0 \leq s_0\sigma \leq \sigma$, which gives both desired conclusions. Since σ is a strategy relative to x, we know that $\langle x^\top s_0| = \langle \sigma|$. We calculate the saturation of $s_0\sigma$

$$\langle x^\top|(s_0\sigma) = r(s_0\sigma x^\top) = \langle x^\top|\langle\sigma|(s_0) = \langle x^\top|\langle x^\top s_0|(s_0) \geq \langle x^\top|(s_0) \geq 1,$$

where we used properties of modalities for the first two steps, then the hypothesis that σ is a strategy. To conclude, we used that $\langle x^\top s_0|(s_0) \geq \langle s_0|(s_0) = s_0$ and that s_0 is a covering set. □

By conversion, we also get $\overline{\sigma}s_0 = s_0$ and $s_0 \leq \overline{\sigma}$. This immediately gives the following properties of a strategy σ relative to a section s_0:

$$\sigma\cdot\sigma = \sigma, \quad \overline{\sigma}\cdot\overline{\sigma} = \overline{\sigma}, \quad \sigma \leq \sigma\cdot\overline{\sigma} \quad \text{and} \quad \overline{\sigma} \leq \sigma\cdot\overline{\sigma}. \tag{8.4.1}$$

Indeed, $\sigma\sigma = \sigma s_0 \sigma = \sigma s_0 = \sigma$ by the fact that $r(\sigma) = s_0$ and Lemma 8.4.3, the case of $\overline{\sigma}$ follows by conversion. Additionally, $s_0 \leq \overline{\sigma}$ so $\sigma = \sigma s_0 \leq \sigma\overline{\sigma}$ and symmetrically for $\overline{\sigma}$.

Next, we will show that the normal forms and exhaustive iteration of a convergent element give us a section and a strategy, respectively. First, we show:

8.4.4. Lemma ([16]). *Let K a Boolean MKA. For a convergent element $x \in K$, we have $|x^\top\rangle = |exh(x)\rangle\langle exh(x)|$.*

Proof. One direction holds since $exh(x)\overline{exh(x)} \leq x^*\overline{x}^* \leq x^\top$ so by monotonicity of taking diamonds and reversal of diamonds by conversion, we get $|x^\top\rangle \geq |exh(x)\rangle\langle exh(x)|$. The other inequality is obtained via the star induction law for modalities (**??**). Indeed, it suffices to prove that

$$|1\rangle + |x + \overline{x}\rangle|exh(x)\overline{exh(x)}\rangle \leq |exh(x)\overline{exh(x)}\rangle.$$

We prove the inequality for each of the summands. We treat the case of $|1\rangle$ first: by definition,

$$|exh(x)\overline{exh(x)}\rangle(p) = d(x^*\neg d(x)r(px^*)) = d(x^*r(px^*)\neg d(x)),$$

where we used the so-called *import-export law* [28] $r(yp) = r(y)p$ for codomains and that multiplication is commutative in K_d. Since $p \leq 1$ we have

$$px^*r(px^*)\neg d(x) \leq x^*r(px^*)\neg d(x),$$

and since $(px^*)r(px^*) = px^*$, applying domain on both sides yields

$$|exh(x)\overline{exh(x)}\rangle(p) \geq d(px^*\neg d(x)) = pd(exh(x)) = p,$$

where we used the import-export law for domains $d(py) = pd(y)$ and Lemma 6.2.7. Thus $|exh(x)\overline{exh(x)}\rangle \geq |1\rangle$. The case of $|x\rangle$ follows by the star unfold axiom:

$$|x\rangle|x^*\neg d(x)\overline{x}^*\rangle = |xx^*\neg d(x)\overline{x}^*\rangle \leq |x^*\neg d(x)\overline{x}^*\rangle.$$

The final case follows by the hypothesis of confluence:

$$\begin{aligned}|\overline{x}\rangle|x^*\neg d(x)\overline{x}^*\rangle = \langle x||x^*\rangle\langle exh(x)| &\leq \langle x^*||x^*\rangle\langle exh(x)| \\ &\leq |x^*\rangle\langle x^*|\langle exh(x)| \\ &\leq |x^*\rangle\langle exh(x)x^*| = |x^*\neg d(x)\overline{x}^*\rangle,\end{aligned}$$

where we also used $exh(x)x^* = exh(x)$. Applying the star induction axiom for modalities, we obtain the result. □

Now we are ready to relate exhaustion and normal forms to strategies and sections, respectively:

8.4.5. Proposition ([16]). *If x is convergent, then $NF(x)$ is a section of x. Furthermore, any skeleton σ of $exh(x)$ is a strategy for x with respect to $NF(x)$, and we have*

$$\sigma \leq NF(x) + x^{+}, \qquad \overline{\sigma} \leq NF(x) + \overline{x}^{+} \qquad \textit{and} \quad \overline{\sigma}\sigma = NF(x)$$

Proof. First we show that $\mathrm{NF}(x)$ is a section. It is a covering set since

$$|x^{\top}\rangle(\mathrm{NF}(x)) \geq |exh(x)\rangle(\mathrm{NF}(x)) = d(exh(x)) = 1$$

where the last step is by Lemma 6.2.7. Suppose now there is some $s \in K_d$ such that $s \leq \mathrm{NF}(x)$ and s is a covering set. Since $s \leq \mathrm{NF}(x) \leq \neg d(x)$, the star unfold and antidomain axioms give $s \cdot exh(x) = s$, so $\langle exh(x)|(s) = s$.

Therefore $1 = |x^{\top}\rangle(s) = |exh(x)\rangle\langle exh(x)|(s) = |exh(x)\rangle(s)$, where we used Lemma 8.4.4. This means that

$$s \geq \langle exh(x)||exh(x)\rangle(s) = \langle exh(x)|(1) = r(exh(a)) = \mathrm{NF}(x),$$

where the first inequality is by Lemma 6.2.7, so we may conclude $\mathrm{NF}(x) = s$, *i.e.* $\mathrm{NF}(x)$ is minimal.

Now we show that a skeleton σ of $exh(x)$ is a strategy for x relative to $\mathrm{NF}(x)$. Note that $|x^{\top}\mathrm{NF}(x)\rangle = |x^{\top}\rangle\langle\mathrm{NF}(x)\rangle$ and $\langle x^{\top}\mathrm{NF}(x)| = \langle\mathrm{NF}(x)\rangle\langle x^{\top}|$. By Lemma 8.4.4,

$$|x^{\top}\mathrm{NF}(x)\rangle = |exh(x)\rangle\langle exh(x)|\langle\mathrm{NF}(x)\rangle = |exh(x)\rangle\langle\mathrm{NF}(x)\rangle = |exh(x)\rangle,$$

since $\mathrm{NF}(x)exh(x) = \mathrm{NF}(x)$, and $exh(x)\mathrm{NF}(x) = exh(x)$. A symmetric proof gives $\langle x^{\top}\mathrm{NF}(x)| = \langle exh(x)|$. Since σ is a skeleton of $exh(x)$, its diamonds coincide with those of $exh(x)$ and so, by what precedes, also with those of $x^{\top}\mathrm{NF}(x)$. Since $exh(x) \leq x^{\top}\mathrm{NF}(x)$, σ is a wide sub of $x^{\top}\mathrm{NF}(x)$. Minimality of σ as a wide sub follows from that same inequality plus the hypothesis that it is a skeleton of $exh(x)$. To conclude, note that $\mathrm{NF}(x)\sigma \leq \mathrm{NF}(x)exh(x) = \mathrm{NF}(x)$. The first inequality follows from

$$\sigma \leq exh(x) = x^{*}\mathrm{NF}(x) = (1 + xx^{*})\mathrm{NF}(x) \leq \mathrm{NF}(x) + xx^{*} = \mathrm{NF}(x) + x^{+},$$

where we used the definition of $exh(x)$, the left star unfold axiom, $\mathrm{NF}(x) \leq 1$ and the definition of the Kleene plus. The inequality for $\overline{\sigma}$ is then obtained by conversion. Finally, since $\sigma \leq exh(x)$ and x is confluent, we get

$$\overline{\sigma}\sigma \leq \overline{exh(x)}exh(x) = \mathrm{NF}(x)\overline{x}^{*}x^{*}\mathrm{NF}(x) \leq \mathrm{NF}(x)x^{*}\overline{x}^{*}\mathrm{NF}(x) = \mathrm{NF}(x),$$

where we also used that $\mathrm{NF}(x) \leq \neg d(x) = \neg r(\overline{x})$. □

8.5. ABSTRACT COHERENCE IN 2-MKA

Now we are equipped to prove the abstract coherence theorem in the context of globular 2-Kleene algebras. First, we prove Theorem 8.5.1, a Kleene algebraic version of the strategic Newman's lemma, Theorem 7.1.22. We then prove Theorem 8.5.2, the main result of this chapter. These results were published in [16].

8.5.1. Theorem (Coherent normalising Newman's lemma [16]). *Let K be a Boolean globular 2-Kleene algebra such that*

- **i)** $(K_0, +, 0, \odot_0, 1_0, \neg_0)$ *is a complete Boolean algebra,*
- **ii)** K_1 *is continuous with respect to 0-restriction, that is for all* $\psi, \psi' \in K_1$ *and* $(p_\alpha)_\alpha \subseteq K_0$ *we have* $\psi \odot_0 \vee p_\alpha \odot_0 \psi' = \vee\, (\psi \odot_0 p_\alpha \odot_0 \psi')$.

Let $\phi \in K_1$ *be convergent and* σ *be a skeleton of* $exh(\phi)$. *If* A *is a local confluence filler for* $(\overline{\phi}, \phi)$, *then* $|\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}) \geq \overline{\phi}^{*_0} \odot_0 \phi^{*_0}$.

Proof. We denote 0-multiplication by juxtaposition. First, we define a predicate RNP expressing restricted normalised paving. Given $p \in K_0$, let

$$RNP(p) \quad \Leftrightarrow \quad |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \geq \overline{\phi}^{*_0} p \phi^{*_0}.$$

By completeness of K_0, we set $r := sup\{\, p \,|\, RNP(p)\}$ and by continuity of restriction we may infer $RNP(r)$. Furthermore, by downward closure of RNP, we have $RNP(p)$ if, and only if, $p \leq r$. We thereby deduce:

$$\begin{aligned} \forall p.\ (RNP(\langle\phi|_0 p) \Rightarrow RNP(p)) &\Leftrightarrow \forall p.\ (\langle\phi|_0 p \leq r \Rightarrow p \leq r) \\ &\Leftrightarrow \forall p.\ (p \leq |\phi]_0 r \Rightarrow p \leq r) \\ &\Leftrightarrow |\phi]_0 r \leq r \end{aligned}$$

where we used the Galois connection (6.1.12). Thus, it suffices to show that

$$\forall p.\ (RNP(\langle\phi|_0 p) \Rightarrow RNP(p))$$

in order to conclude that $r = 1_0$, by Noethericity of ϕ. This method constitutes formalised Noetherian induction for Boolean MKA.

Given $p \in K_0$, we denote by p_ϕ the element $\langle\phi|_0(p) = |\overline{\phi}\rangle_0(p)$. We have

$$p\phi = p\phi r_0(p\phi) = p\phi\langle\phi|_0(p) \leq \phi p_\phi,$$

and similarly $\overline{\phi}p \leq p_\phi\overline{\phi}$. Using the star unfold axioms, we thereby deduce that

$$\overline{\phi}^{*_0} p \phi^{*_0} \leq \overline{\phi}^{*_0} p + \overline{\phi}^{*_0}\overline{\phi} p \phi \phi^{*_0} + p\phi^{*_0} \leq \overline{\phi}^{*_0} p + \overline{\phi}^{*_0} p_\phi \overline{\phi} \phi p_\phi \phi^{*_0} + p\phi^{*_0}.$$

We first examine the middle summand:

$$\begin{aligned} &\overline{\phi}^{*_0} p_\phi \overline{\phi} \phi p_\phi \phi^{*_0} \\ &\quad \leq \overline{\phi}^{*_0} p_\phi |A\rangle_1(\phi^{*_0}\overline{\phi}^{*_0}) p_\phi \phi^{*_0} \\ &\quad \leq |\overline{\phi}^{*_0} p_\phi A p_\phi \phi^{*_0}\rangle_1(\overline{\phi}^{*_0} p_\phi \phi^{*_0} \overline{\phi}^{*_0} p_\phi \phi^{*_0}) \\ &\quad \leq |\hat{A}\rangle_1(\overline{\phi}^{*_0} p_\phi \phi^{*_0} \overline{\phi}^{*_0} p_\phi \phi^{*_0}) \\ &\quad \leq |\hat{A}\rangle_1(|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})\overline{\phi}^{*_0} p_\phi \phi^{*_0}) \\ &\quad \leq |\hat{A}\rangle_1(|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}\overline{\phi}^{*_0} p_\phi \phi^{*_0})) \\ &\quad \leq |\hat{A}\rangle_1(|\hat{A}^{*_1}\rangle_1(|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}\sigma\overline{\sigma}))) \\ &\quad \leq |\hat{A} \odot_1 \hat{A}^{*_1} \odot_1 \hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}). \end{aligned}$$

$$\begin{tikzcd}
 & & p \arrow[dr, "\phi"] & & \\
 & p_\phi \arrow[ur, "\overline{\phi}"] \arrow[dr, dashed, "\phi^*"] & & p_\phi \arrow[dr, "\phi^*"] & \\
\cdot \arrow[ur, "\overline{\phi}^*"] \arrow[dr, "\sigma"'] & & \cdot \arrow[ur, dashed, "\overline{\phi}^*"] \arrow[dr, "\sigma"] & & \cdot \\
 & \cdot \arrow[ur, "\overline{\sigma}"] \arrow[rr, equal] & & \cdot \arrow[ur, "\overline{\sigma}"'] &
\end{tikzcd}$$

where we used that A is a local confluence filler for the first step, then commutation of modalities with whiskering (8.3.1) and the definition of $\hat{A}$ (8.3.2) for the second and third steps. We then use the induction hypothesis $RPN(p_\phi)$ on the left instance of $\overline{\phi}^{*_0} p_\phi \phi^{*_0}$, followed by commutation of modalities with whiskering and whisker absorption (8.3.3), and then repeat for the instance on the right. Finally, we used that $\hat{A} \odot_1 \hat{A}^{*_1} \leq \hat{A}^{*_1} \odot_1 \hat{A}^{*_1} \leq \hat{A}^{*_1}$, monotonicity of taking diamonds and $\overline{\sigma}\sigma = \mathrm{NF}(\phi) = r(\sigma)$, a consequence of Proposition 8.4.5.

It remains to show that $\overline{\phi}^{*_0} p, p\phi^{*_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})$. First, observe that we have

$$\begin{aligned}
\overline{\sigma} p \phi^{*_0} &= \overline{\sigma} p + \overline{\sigma} p \phi^{+_0} \\
&\leq \overline{\sigma} + (\mathrm{NF}(\phi) + \overline{\phi}^{+_0}) p \phi^{+_0} \\
&= \overline{\sigma} + \overline{\phi}^{+_0} p \phi^{+_0} \leq \sigma\overline{\sigma} + \overline{\phi}^{+_0} p \phi^{+_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).
\end{aligned}$$

$$\begin{tikzcd}
 & p \arrow[dr, "\phi^*"] & \\
\cdot \arrow[ur, "\overline{\sigma}"] \arrow[dr, "\sigma"'] & & \cdot \\
 & \cdot \arrow[ur, "\overline{\sigma}"'] &
\end{tikzcd}$$

The first step is by the unfold axiom, the second uses Proposition 8.4.5 to bound $\overline{\sigma}$. The third step uses the fact that $\mathrm{NF}(\phi)$ is a left annihilator for ϕ^{+_0} since by definition we have $\mathrm{NF}(\phi) \leq \neg d_0(\phi)$. Finally we use the fact that $\overline{\sigma} \leq \sigma\overline{\sigma}$ (8.4.1) coupled with $id_{K_1} = |1_1\rangle_1 \leq |\hat{A}^{*_1}\rangle_1$, *i.e.* reflexivity of $\hat{A}^{*_1}$, as well as the bound established by the previous calculation.

For convergent ϕ, we have $d_0(exh(\phi)) = d_0(\phi^{*_0}\neg d_0(\phi)) = 1_0$ by Lemma 6.2.7. Since σ is a skeleton of $exh(\phi)$, we have $d_0(\sigma) = 1_0$. By the converse axiom (6.1.20), this means that $\sigma\overline{\sigma} \geq 1_0$. Therefore,

$$\begin{aligned}
p\phi^{*_0} &\leq \sigma\overline{\sigma} p \phi^{*_0} \\
&\leq \sigma |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \\
&\leq |\hat{A}^{*_1}\rangle_1(\sigma\sigma\overline{\sigma}) = |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}),
\end{aligned}$$

$$\begin{tikzcd}
\cdot \arrow[dr, "\sigma"'] & & p \arrow[dr, "\phi^*"] & \\
 & \cdot \arrow[ur, "\overline{\sigma}"] \arrow[dr, "\sigma"'] & & \cdot \\
 & & \cdot \arrow[ur, "\overline{\sigma}"'] &
\end{tikzcd}$$

where we used commutation of whisker with modalities and whisker absorption, as well as $\sigma\sigma = \sigma$ (8.4.1). A symmetric argument yields $\overline{\phi}^{*_0} p \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})$, concluding the proof. □

The use of formalised Noetherian induction, as well as the calculation establishing the upper bound for the middle summand, are similar to those in the proof of Newman's lemma in [28], reproduced in Section 6.2 as Theorem 6.2.9. Due to the fact that our result involves confluences in σ, the bounds for the outer summands require a different approach.

As a direct consequence of Theorem 8.5.1, we obtain the following result, which formalises Theorem 8.5.2. Indeed, if (Φ, X) is a $(2,0)$-polygraph satisfying the corresponding hypotheses, Theorem 8.5.2 lifts the result to the power set when applied to $\phi := \{1_x \mid x \in \Phi_1\} \cup \{1_{1_a} \mid a \in \Phi_0\}$ and $A = X$, viewed as elements of $K(\Phi, X)$. Following the argument given in Section 8.3, the conclusion asserts that *for every* zig-zag sequence $f : a \to b \in \Phi_1^\top$, *there exists* a 2-cell $\alpha_f : f \Rightarrow \sigma_a \star_0 \sigma_b^-$ obtained by whiskering and composing elements of X. In a 2-groupoid, this is equivalent to the existence of a 2-cell $f \star_0 \sigma_b \Rightarrow \sigma_a$.

8.5.2. Theorem (Abstract coherence theorem [16]). *Let K be a Boolean globular 2-Kleene algebra satisfying the additional hypotheses in Theorem 8.5.1 and $\phi \in K_1$ convergent. Given a normalisation strategy σ and a local confluence filler A for $(\overline{\phi}, \phi)$, we have*

$$|\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}) \geq \phi^{\top_0} = (\phi + \overline{\phi})^{*_0}.$$

Proof. We denote 0-multiplication by juxtaposition. As a result of Theorem 8.5.1 we have $|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \geq \overline{\phi}^{*_0}\phi^{*_0}$. By the star induction axiom, it suffices to show:

$$1_0 + (\phi + \overline{\phi})|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).$$

By (6.1.20) and Proposition 8.4.5, we have $\sigma\overline{\sigma} \geq d_0(\sigma) = 1_0$, so by reflexivity of $\hat{A}^{*_1}$, *i.e.* $1_1 \leq \hat{A}^{*_1}$, we have $1_0 \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})$. Furthermore, since $\phi \leq \overline{\phi}^{*_0}\phi^{*_0}$ we have:

$$\phi|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq \overline{\phi}^{*_0}\phi^{*_0}|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).$$

The case of $\overline{\phi}$ is identical. We conclude via the star induction axiom. □

Note that in essence, the proof of this coherence theorem is similar to the proof of that of Theorem 7.1.16 found in Section 7.1.21. Indeed, first we prove a strategic Newman's lemma, and then we apply a Church-Rosser argument with respect to confluences in the strategy.

CHAPTER 9.
HIGHER KLEENE ALGEBRAS

In this chapter, we address rewriting paradigms with Kleene algebra in more generality than in the previous. Mainly, we introduce higher Kleene algebras (HKA), generalising the case of globular 2-Kleene algebras to arbitrary dimension. We provide a full account of their properties, both as algebraic structures and as algebraic tools for coherence proofs. Further, we prove coherent, higher versions of Newman's lemma and the Church-Rosser theorem in this context, and express the coherence theorem, Theorem 8.5.2, in this higher setting. Finally, we show that the Kleene algebraic versions of coherent rewriting theorems correspond to their polygraphic counterparts through the polygraphic model of HKA.

In Section 9.1, we introduce globular n-Kleene algebras step by step, starting with n-dioids and progressively adding more structure. In particular, Section 9.1.19 provides a full description of the HKA obtained by lifting higher polygraphs to the power-set level. Next, in Section 9.2 we describe coherent confluence properties in HKA, and prove two versions of the Church-Rosser theorem, Theorems 9.2.8 and 9.2.9, the first using a external induction technique, the other using the internal induction principle provided by the Kleene star axioms. Section 9.3 contains the HKA version of Newman's lemma, Theorem 9.3.2. In Section 9.4, we see the coherence theorem for HKA. Finally, Theorems 9.2.9 and 9.3.2 are related to their polygraphic counterparts via the power-set model in Section 9.5.

Results in this chapter are original contributions, first appearing in [17], excepting Proposition 9.1.18 which appears for the first time in this thesis.

9.1. HIGHER KLEENE ALGEBRAS

9.1.1. n-Dioid. We define a 0-*dioid* as a bounded distributive lattice and a 1-*dioid* as a dioid. More generally, for $n \geq 1$, an n-*dioid* is a structure $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$ satisfying the following conditions:

i) $(S, +, 0, \odot_i, 1_i)$ is a dioid for $0 \leq i < n$,

ii) the following *lax interchange laws* hold for all $0 \leq i < j < n$:

$$(x \odot_j x') \odot_i (y \odot_j y') \leq (x \odot_i y) \odot_j (x' \odot_i y') \tag{9.1.2}$$

iii) Higher dimensional units are idempotents of lower dimensional multiplications, *i.e.*

$$1_j \odot_i 1_j = 1_j \tag{9.1.3}$$

for $0 \leq i < j < n$.

With lax interchange laws, by contrast to the equational case, we need not worry about an Eckmann-Hilton collapse.

9.1.4. Domain n-semirings. For $n = 0$, we stipulate that a *domain* 0*-semiring* is a 0-diod. For $n \geq 1$, a *domain* n*-semiring* is an n-dioid $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$ equipped with n domain maps $d_i : S \to S$, for all $0 \leq i < n$, satisfying the following conditions:

i) $(S, +, 0, \odot_i, 1_i, d_i)$ is a domain semiring,

ii) $d_{i+1} \circ d_i = d_i$.

For $0 \leq i < n$, the set $S_{d_i} = d_i(S)$ will be called the i*-dimensional domain algebra*, and will be denoted by S_i. Furthermore, to distinguish elements of distinct dimensions $0 \leq i < j < n$, we henceforth denote elements of S_i by $p, q, r, \ldots$, elements of S_j by $\phi, \psi, \xi, \ldots$, and other elements of S by $A, B, C, \ldots$ This notation simplifies the reading of proofs when elements of different dimensions are interacting. For a natural number $k \geq 0$, the k*-fold* i*-multiplication* of an element A of S, for $0 \leq i < n$, is defined by

$$A^{0_i} = 1_i, \qquad A^{k_i} = A \odot_i A^{(k-1)_i}.$$

The axioms **ii)** and **iii)** from Section 9.1.1 for n-dioids provide the basic algebraic structure for reasoning about higher-dimensional rewriting systems. Indeed, the dependencies between multiplications of different dimensions expressed by the lax interchange laws capture the lifting of the equational interchange law for n-categories, while the idempotence of i-multiplication for the j-unit expresses completeness of the set of j-dimensional cells in an n-category with respect to i-composition. In this way, these axioms begin to capture the higher dimensional character of polygraphs, as is made clear in Section 9.1.19, in which we provide a model of this structure based on polygraphs. The domain axiom **ii)** from Section 9.1.4 further captures characteristics of dimension, which are expressed abstractly in the following proposition.

9.1.5. Proposition. *For $n \geq 1$, in any domain n-semiring S, for all $0 \leq i < j < n$, the following conditions hold:*

i) $d_j \circ d_i = d_i$,

ii) $d_j(1_i) = 1_i$,

iii) $1_i \leq 1_j$,

iv) $S_i \subseteq S_j$,

v) $(S_j, +, 0, \odot_i, 1_i, d_i)$ *is a domain sub semiring of* $(S, +, 0, \odot_i, 1_i, d_i)$ *and* $d_i(S_j) = S_i$,

vi) $(S_j, +, 0, \odot_k, 1_k, d_k)_{0\leq k\leq i}$ *is a domain sub* $(i+1)$*-semiring of* $(S, +, 0, \odot_k, 1_i, d_k)_{0\leq k\leq i}$.

vii) $(S_i, +, 0, \odot_i, 1_i)$ *is a* 0*-dioid.*

Proof. The first identity is proved by a simple induction on axiom **ii)** in (9.1.4). The second one quickly follows, since $d_i(1_i) = 1_i$ follows from the domain semiring axioms, and thus $d_j(1_i) = 1_i$ using **i)**. The third identity is again a direct consequence, since by **ii)** we know that $1_i \in S_j$, and that 1_j is the greatest element of S_j. The fourth one follows since $x \in S_i$ if, and only if, $d_i(x) = x$, which is equivalent to $d_j(x) = x$ by **ii)**. The fifth identity is verified by noticing that the inclusion $S_j \hookrightarrow S$ is a morphism of domain semirings with the operation $\odot_i$. Furthermore, since $d_i(S_j) \subseteq S_i$ and $S_i \subseteq S_j$, we have $d_i(S_j) = S_i$. Noticing that, in fact, $S_j \hookrightarrow S$ is a morphism of domain semirings with the operation $\odot_k$ for any $0 \leq k \leq i$ gives us **vi)**. The final result follows from basic properties of domain semirings. □

Given an n-semiring S, we denote by S^{op} the n-semiring in which the order of each multiplication operation has been reversed. An n-semiring S is a *codomain n-semiring* if S^{op} is a domain n-semiring. The codomain operators are denoted by r_i. A *modal n-semiring* is an n-semiring with domains and codomains, in which the coherence conditions $d_i \circ r_i = r_i$ and $r_i \circ d_i = d_i$ hold for all $0 \leq i < n$.

9.1.6. Remarks. Section 6.1.22 recalls that the path algebra $K(P)$ defined as the power set of 1-cells in the free category generated by a 1-polygraph $P = (P_0, P_1)$ is a model of modal 1-semiring. The domain algebra $K(P)_d$ is isomorphic to the power set of P_0. As recalled in Section 6.1.3, in the general case of a domain semiring $(S, +, 0, \cdot, 1, d)$, the domain algebra S_d forms a bounded distributive lattice with $+$ as join, $\cdot$ as meet, 0 as bottom and 1 as top. It is for this reason that we consider a 0-dioid as a bounded distributive lattice. Indeed, the idempotence and commutativity of the multiplication operation simulate the properties of a set of identity 1-cells.

Note also that, in Section 9.1.19, we will construct higher-dimensional path algebras over n-polygraphs and show that these form models of modal n-semirings. In this case it makes sense that $(S_i, +, 0, \odot_i, 1_i)$ is a 0-dioid, since an i-cell $f : u \to v$ of an n-category $\mathcal{C}$ is a 0-cell in the hom-category $\mathcal{C}(u, v)$.

9.1.7. Diamond operators. Let S be a modal n-semiring. We introduce *forward* and *backward i-diamond* operators defined via (co-)domain operators in each dimension by analogy to (6.1.5). For any $0 \leq i < n$, $A \in S$ and $\phi \in S_i$, we define

$$|A\rangle_i(\phi) = d_i(A \odot_i \phi), \quad \text{and} \quad \langle A|_i(\phi) = r_i(\phi \odot_i A). \tag{9.1.8}$$

In the absence of antidomains, box operators cannot be expressed in this setting. These diamond operators have all of the properties recalled in Section 6.1.8 with respect to i-multiplication and elements of S_i.

9.1.9. p-Boolean domain semirings. For $0 \leq p < n$, a domain n-semiring $(S, +, 0, \odot_i, 1_i, d_i)_{0 \leq i < n}$ is called *p-Boolean* if it is augmented with $(p+1)$ maps

$$(ad_i : S \to S)_{0 \leq i \leq p}$$

such that for all $0 \leq i \leq p$, the following conditions are satisfied:

- **i)** $(S, +, 0, \odot_i, 1_i, ad_i)$ is a Boolean domain semiring,
- **ii)** $d_i = ad_i^2$.

By definition, a 0-Boolean domain 1-semiring is a Boolean domain semiring, and by convention we define a *0-Boolean domain 0-semiring* as a Boolean algebra. We define a *p-Boolean codomain semiring* as an n-semiring such that its opposite n-semiring is a p-semiring with antidomains. In this case the anticodomain operators are denoted ar_i. We then obtain a notion of p-Boolean modal n-Kleene algebra. These are modal n-Kleene algebras which are both p-Boolean with respect to domain and codomain.

9.1.10. Remark. The key difference between modal n-semirings and their p-Boolean counterparts is that the latter are equipped with negation operations in their lower dimensions. Indeed, in a p-Boolean modal Kleene algebra K, for every $0 \leq i \leq p$, the tuple

$$(K_i, +, 0, \odot_i, 1_i, ad_i)$$

is a Boolean algebra. For this reason, we denote the restriction of ad_i to K_i by $\neg_i$. Furthermore, as recalled in (6.1.8), for $0 \leq j \leq p$, $A \in K$ and $\phi \in K_j$ we can define *forward* (resp. *backward*) *box operators*

$$|A]_j(\phi) := \neg_j(|A\rangle_j(\neg_j\phi)) \qquad (\textit{resp. } [A|_j(\phi) := \neg_j(\langle A|_j(\neg_j\phi)))$$

9.1.11. Globular modal n-semiring. A modal semiring S is called *globular* if the following *globular relations* hold for $0 \leq i < j < n$ and $A, B \in K$:

$$\begin{array}{llll} d_i \circ d_j = d_i \ \text{ and } \ d_i \circ r_j = d_i, & (9.1.12) & d_j(A \odot_i B) = d_j(A) \odot_i d_j(B), & (9.1.14) \\ r_i \circ d_j = r_i, \ \text{ and } \ r_i \circ r_j = r_i, & (9.1.13) & r_j(A \odot_i B) = r_j(A) \odot_i r_j(B). & (9.1.15) \end{array}$$

The intuition here is that A is a *collection* of globular cells and that for $k \in \{i, j\}$, $d_k(A)$ (resp. $r_k(A)$) is a *collection* of globular k-cells each of which is the k-source (resp. k-target) of some cell belonging to A. In Section 9.1.19 this intuition is elucidated via the polygraphic model.

An element A of S will be represented graphically by the following diagram with respect to its i- and j-borders, when $i < j$:

$$\begin{tikzcd} d_i(A) \arrow[rr, bend left, "d_j(A)"] \arrow[rr, bend right, "r_j(A)"'] & \Downarrow A & r_i(A) \end{tikzcd}$$

Below are graphical representations of i- and j-multiplication with respect to i- and j-borders. The illustrations underline the fact that multiplication of elements in a Kleene algebra is equivalent to multiplying their restrictions to the appropriate domain or range, as below:

$$A \odot_i B = (A \odot_i r_i(A)) \odot_i (d_i(B) \odot_i B) = (A \odot_i d_i(B)) \odot_i (r_i(A) \odot_i B),$$

where we have used properties of domain semirings given in (6.1.3), and that these restrictions are compatible with the globular relations.

$$\begin{tikzcd}
d_i(A\odot_i d_i(B)) \arrow[r, bend left=50, "d_j(A\odot_i d_i(B))"] \arrow[r, bend right=50, "r_j(A\odot_i d_i(B))"'] \arrow[r, phantom, "\Downarrow A\odot_i d_i(B)"] & r_i(A)\odot_i d_i(B) \arrow[r, bend left=50, "d_j(r_i(A)\odot_i B)"] \arrow[r, bend right=50, "r_j(r_i(A)\odot_i B)"'] \arrow[r, phantom, "\Downarrow r_i(A)\odot_i B"] & r_i(r_i(A)\odot_i B) & \equiv & d_i(A\odot_i B) \arrow[r, bend left=50, "d_j(A\odot_i B)"] \arrow[r, bend right=50, "r_j(A\odot_i B)"'] \arrow[r, phantom, "\Downarrow A\odot_i B"] & r_i(A\odot_i B)
\end{tikzcd}$$

$$\begin{tikzcd}
d_i(A)\odot_i d_i(B) \arrow[r, bend left=50, "d_j(A\odot_j d_j(B))"] \arrow[r, "r_j(A)\odot_j d_j(B)" description] \arrow[r, bend right=50, "r_j(r_j(A)\odot_j B)"'] \arrow[r, phantom, shift left=4ex, "\Downarrow A\odot_j d_j(B)"] \arrow[r, phantom, shift right=4ex, "\Downarrow r_j(A)\odot_j B"] & r_i(A)\odot_i r_i(B) & \equiv & d_i(A\odot_j B) \arrow[r, bend left=40, "d_j(A\odot_j B)"] \arrow[r, bend right=40, "r_j(A\odot_j B)"'] \arrow[r, phantom, "\Downarrow A\odot_j B"] & r_i(A\odot_j B)
\end{tikzcd}$$

9.1.16. Modal n-Kleene algebra. An *n-Kleene algebra* is an n-dioid K equipped with operations $(-)^{*_i} : K \to K$ satisfying the following conditions:

- **i)** $(K, +, 0, \odot_i, 1_i, (-)^{*_i})$ is a Kleene algebra for $0 \leq i < n$,
- **ii)** For $0 \leq i < j < n$, the Kleene star operation $(-)^{*_j}$ is a *lax morphism* with respect to the i-whiskering of j-dimensional elements on the right (resp. left), that is for all $A \in K$ and $\phi \in K_j$,

$$\phi \odot_i A^{*_j} \leq (\phi \odot_i A)^{*_j}, \qquad \text{and} \qquad (\text{resp. } A^{*_j} \odot_i \phi \leq (A \odot_i \phi)^{*_j}). \qquad (9.1.17)$$

Note however, that as a result of the interchange law and the induction axioms, we have the following result. This appears for the first time in this thesis, as it was noticed when considering higher quantalic structures, see Chapter 10.

9.1.18. Proposition. *Let K be a n-Kleene algebra and $A, B \in K$. Then*

$$(A \odot_j B)^{*_i} \leq A^{*_i} \odot_j B^{*_i}.$$

Proof. We use the induction axiom, see Section 6.1.13, with respect to i-multiplication. We have

$$\begin{aligned} 1_i + (A \odot_j B) \odot_i (A^{*_i} \odot_j B^{*_i}) &\leq 1_i + (A \odot_i A^{*_i}) \odot_j (B \odot_i B^{*_i}) \\ &\leq 1_i + A^{*_i} \odot_j B^{*_i} \leq A^{*_i} \odot_j B^{*_i}, \end{aligned}$$

where thee first inequality uses the interchange law, the second the unfold law for the i-star, and finally the fact that $1_i \leq A^{*_i}, B^{*_i}$ and $1_i \odot_j 1_i = 1_i$.

By the induction axiom with respect to the i-multiplication, we obtain the desired inequality. □

As in the case of 1-Kleene algebras, recalled in (6.1.13), the notions of (p-Boolean) n-semiring structures with (co)domains are compatible with the notion of n-Kleene algebra. Hence, an *n-Kleene algebra with domains* (resp. *codomains*) is a n-Kleene algebra such that the underlying semiring has domains (resp. codomains). When the underlying n-semiring is modal, we have a *modal n-Kleene algebra*. If it is p-Boolean, we have a *p-Boolean modal n-Kleene algebra*. We say that these are *globular* when the underlying modal n-semiring is globular. We also refer to these structures as *higher Kleene algebras*, or HKA for short.

We also remark that this definition of higher Kleene algebras does not necessitate stopping at a finite number n. Indeed, we could consider ω-Kleene algebras along the same lines as above. Finally, note that for $n = 2$, we obtain the standard concurrent Kleene algebra axioms [75], except that $1_0 = 1_1$ is normally assumed in this case.

9.1.19. A polygraphic model of higher Kleene algebras. Let P be an n-polygraph and Γ a cellular extension of the free $(n, n-1)$-category $P_n^\top$. We define an $(n+1)$-modal Kleene algebra $K(P,\Gamma)$, the *full* $(n+1)$*-path algebra* over $P_n^\top[\Gamma]$, as follows:

i) The carrier set of $K(P,\Gamma)$ is the power set $\mathcal{P}(P_n^\top[\Gamma])$, whose elements, denoted by $A, B, C, \ldots$, are sets of $(n+1)$-cells. We denote these $(n+1)$-cells by $\alpha, \beta, \gamma, \ldots$ in what follows.

ii) Recall that for α a k-cell, the elements $s_i(\alpha)$, $t_i(\alpha)$, $\iota_k^l(\alpha)$ were defined for $0 \leq i \leq k \leq l \leq n+1$ in Sections 5.4.1 and 5.4.6. When $k \leq i$, we define $s_i(\alpha) = t_i(\alpha) = \iota_k^i(\alpha)$,

iii) Recall that the i-composition of a k-cell α and an l-cell β for $0 \leq i < k \leq l \leq n+1$ was defined in Sections 5.4.1 and 5.4.6. For $0 \leq k \leq l < n+1$, we define

$$\alpha \star_i \beta = \begin{cases} \iota_k^{i+1}(\alpha) \star_i \beta \text{ for } k \leq i < l, \\ \iota_k^{i+1}(\alpha) \star_i \iota_l^{i+1}(\beta) \text{ for } l \leq i. \end{cases}$$

iv) For $0 \leq i < n+1$, the binary operation $\odot_i$ on $K(P,\Gamma)$ corresponds to the lifting of the composition operations of $P_n^\top[\Gamma]$ to the power-set, *i.e.* for any $A, B \in K(P,\Gamma)$,

$$A \odot_i B := \{\alpha \star_i \beta \mid \alpha \in A \wedge \ \beta \in B \wedge t_i(\alpha) = s_i(\beta)\}.$$

v) For $0 \leq i < n+1$, denote by $\mathbb{1}_i$ the set

$$\mathbb{1}_i \;=\; \{\iota_i^{n+1}(u) \mid u \in P_n^{\top}[\Gamma]_i\}.$$

These sets are the units for the multiplication operations, that is we have

$$A \odot_i \mathbb{1}_i \;=\; \mathbb{1}_i \odot_i A \;=\; A.$$

Furthermore, when $i < j$, the inclusion $\mathbb{1}_i \subseteq \mathbb{1}_j$ holds. Indeed, in that case $\iota_i^{n+1}(u) = \iota_j^{n+1}(\iota_i^j(u))$ by uniqueness of identity cells, and $\iota_i^j(u) \in P_n^{\top}(\Gamma)_j$ is a j-cell.

vi) The addition in $K(P,\Gamma)$ is given by set union $\cup$. The ordering is therefore given by set inclusion.

vii) The i-domain and i-codomain maps d_i and r_i are defined by

$$d_i(A) := \{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\}, \qquad \text{and} \qquad r_i(A) := \{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\}.$$

These are thus given by lifting the source and target maps of $P_n^{\top}[\Gamma]$ to the power set. The i-antidomain and i-anticodomain maps are then given by complementation with respect to the set of i-cells:

$$ad_i(A) := \mathbb{1}_i \backslash \{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\}, \qquad \text{and} \qquad ar_i(A) := \mathbb{1}_i \backslash \{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\}.$$

viii) The i-star is given by

$$A^{*_i} = \bigcup_{k \in \mathbb{N}} A^{k_i},$$

where in the above, $A^{0_i} := \mathbb{1}_i$ and $A^{k_i} := A \odot_i A^{(k-1)_i}$.

9.1.20. Proposition ([17]). *For any n-polygraph P and cellular extension Γ of $P_n^{\top}$, $K(P,\Gamma)$ is an n-Boolean $(n+1)$-modal Kleene algebra.*

Additionally, the set Γ^c of rewriting steps generated by Γ as defined in Remark 5.4.9, is represented in n-Kleene algebra by

$$\Gamma^c = 1_n \odot_{n-1} (\cdots \odot_2 (1_2 \odot_1 (1_1 \odot_0 \Gamma \odot_0 1_1) \odot_1 1_2) \odot_2 \cdots) \odot_{n-1} 1_n.$$

Therefore, α is an $(n+1)$-cell of $P_n^{\top}[\Gamma]$ if, and only if, $\alpha \in (\Gamma^c)^{*_n}$.

Proof. It is easy to check that, for $0 \leq i < n+1$, the tuple $(\mathcal{P}((P_n^{\top}[\Gamma])_{n+1}), \cup, \emptyset, \odot_i, \mathbb{1}_i, (-)^{*_i}, d_i, r_i)$ is a modal semiring. The fact that it is n-Boolean is a result of it being a power-set algebra.

Let $A, A', B, B' \in K(P,\Gamma)$ and $0 \leq i < j < n+1$. We want to show that the lax interchange law holds, *i.e.*

$$(A \odot_j B) \odot_i (A' \odot_j B') \subseteq (A \odot_i A') \odot_j (B \odot_i B'). \tag{9.1.21}$$

This is the case since, given $(n+1)$-cells $\alpha \in A, \alpha' \in A', \beta \in B, \beta' \in B'$, if $(\alpha \star_j \beta) \star_i (\alpha' \star_j \beta')$ is defined, then as a consequence of the exchange law for $(n+1)$ categories, we have

$$(\alpha \star_j \beta) \star_i (\alpha' \star_j \beta') = (\alpha \star_i \alpha') \star_j (\beta \star_i \beta') \in (A \odot_i A') \odot_j (B \odot_i B')$$

which gives the desired inclusion (9.1.21). This situation is illustrated by the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta"] & \cdot \arrow[r] \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow \alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow \beta'"] & \cdot
\end{tikzcd}$$

The lax interchange law is not reduced to an equality due to composition of diagrams of the following shape:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[rr, bend left=50] \arrow[r, bend right=60] \arrow[rr, phantom, shift left=4, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta"] & \cdot \arrow[r] \arrow[rr, bend right=50] \arrow[rr, phantom, shift right=4, "\Downarrow \beta'"] & \cdot \arrow[r] \arrow[r, bend left=60] \arrow[r, phantom, shift left=3, "\Downarrow \alpha'"] & \cdot
\end{tikzcd}$$

where $\alpha \in A, \alpha' \in A', \beta \in B, \beta' \in B'$. Indeed, the composition $(\alpha \star_i \alpha') \star_j (\beta \star_i \beta') \in (A \odot_i A') \odot_j (B \odot_i B')$ is defined, whereas neither α and β nor α' and β' are j-composable, meaning that in general the inclusion (9.1.21) is strict.

Further, given $0 \leq i < j < n+1$, we have $\mathbb{1}_j \subseteq \mathbb{1}_j \odot_i \mathbb{1}_j$. Indeed, for any j-cell α, we have $\alpha \star_i \iota_i^{n+1}(t_i(\alpha)) = \alpha$ because $\iota_i^{n+1}(t_i(\alpha))$ is the $(n+1)$-dimensional identity cell on the i-dimensional target of α. Furthermore, $\iota_i^{n+1}(t_i(\alpha)) \in \mathbb{1}_i \subseteq \mathbb{1}_j$, proving the inclusion. Thus $\mathbb{1}_j = \mathbb{1}_j \odot_i \mathbb{1}_j$ since $(P_n^\top(\Gamma))_j$ is closed under i-composition.

Given $0 \leq i < n$, we have $d_{i+1} \circ d_i = d_i$ since the $(i+1)$-dimensional border of an identity cell on an i-cell u is u itself. Since $d_i(\mathbb{1}_i) = \mathbb{1}_i$, we equally have $d_{i+1}(\mathbb{1}_i) = \mathbb{1}_i$.

The first two globularity axioms are immediate consequences of the globularity conditions on the source and target maps of $P_n^\top(\Gamma)$. Furthermore, for $0 \leq i < j < n+1$ and $A, B \in K(P, \Gamma)$, we have $u \in d_j(A \odot_i B)$ if, and only if, there exist $\alpha \in A$ and $\beta \in B$ such that $u = s_j(\alpha \star_i \beta) = s_j(\alpha) \star_i s_j(\beta)$, which is equivalent to $u \in d_j(A) \odot_i d_j(B)$. Similarly, we show that $r_j(A \star_i B) = r_j(A) \odot_i r_j(B)$.

Finally, we consider the Kleene star axioms. It is easy to check that, given a family $(B_k)_{k \in I}$ of elements of $K(P, \Gamma)$ and another element A, we have, for all $0 \leq i < n+1$,

$$A \odot_i \left(\bigcup_{k \in I} B_k \right) = \bigcup_{k \in I} (A \odot_i B_k) \quad \text{ and } \quad \left(\bigcup_{k \in I} B_k \right) \odot_i A = \bigcup_{k \in I} (B_k \odot_i A) .$$

It then follows by routine calculations that the element A^{*_i} defined above satisfies, for each i, the Kleene star axioms, recalled in (6.1.13). It only remains to check that for $0 \leq i < j < n+1$, the j-star is a lax morphism for i-whiskering of j-dimensional elements on the left (the right case being symmetric), that is $\phi \odot_i A^{*_j} \subseteq (\phi \odot_i A)^{*_j}$ for $\phi \in K(P,\Gamma)_j$ and $A \in K(P,\Gamma)$. By construction, $K(P,\Gamma)_j$ is in bijective correspondence with $(P_n^\top(\Gamma))_j$, the set of j-cells of $P_n^\top[\Gamma]$. Considering such elements ϕ and A, we have $\beta \in \phi \odot_i A^{*_j}$ in the following two cases:

i) There exist $u \in \phi$ and $\alpha \in A^{+_j}$, where we recall that $A^{+_j} := A \odot_j A^{*_j}$ is the Kleene plus operation, such that $\beta = u \star_i \alpha$. Since $\alpha \in A^{+_j}$, there exist a $k > 0$ and cells $\alpha_1, \alpha_2, \ldots, \alpha_k \in A$ such that

$$\alpha = \alpha_1 \star_j \alpha_2 \star_j \cdots \star_j \alpha_k.$$

Since $i < j$, the following is a consequence of the exchange law for n-categories:

$$u \star_i (\alpha_1 \star_j \alpha_2 \star_j \cdots \star_j \alpha_k) = (u \star_i \alpha_1) \star_j (u \star_i \alpha_2) \star_j \cdots \star_j (u \star_i \alpha_k),$$

and thus we have $\beta \in (\phi \odot_i A)^{+_j}$.

ii) There exist $u \in \phi$, and $v \in (P_n^\top(\Gamma))_j$ with $v \notin A$ such that $\beta = u \star_i v$. This is due to the fact that $A^{*_j} = \mathbb{1}_j + A^{+_j}$. In that case, we have $\beta \in (P_n^\top(\Gamma))_j$, *i.e.* $\beta \in \mathbb{1}_j$. By the unfold axiom, we have $\mathbb{1}_j \subseteq (\phi \odot_i A)^{*_j}$, and thus $\beta \in (\phi \odot_i A)^{*_j}$.

The fact that α is an $(n+1)$-cell of $P_n^\top[\Gamma]$ if, and only if, $\alpha \in (\Gamma^c)^{*_n}$, follows by definition of Γ^c and the fact that any $(n+1)$-cell of $P_n^\top[\Gamma]$ is an n-composition of rewriting steps. □

9.2. A COHERENT CHURCH-ROSSER THEOREM

Let K be a globular n-modal Kleene algebra and $0 \leq i < j < n$. Before defining fillers in globular modal n-Kleene algebras, we first recall the intuition behind the forward diamond operators, defined in Section 9.1.7. Given $A \in K$ and $\phi, \phi' \in K_j$, recall that by definition

$$|A\rangle_j(\phi) \geq \phi' = d_j(A \odot_j \phi) \geq \phi'.$$

In terms of quantification over sets of cells, as for example in the polygraphic model, this signifies that *for every* element u of ϕ', *there exist* elements v of ϕ and α of A such that the j-source (resp. j-target) of α is u (resp. v). This observation motivates the definitions in the following paragraph.

9.2.1. Confluence fillers. Given elements ϕ and ψ of K_j, we say that an element A in K is a

i) *local i-confluence filler* for (ϕ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq \phi \odot_i \psi,$$

ii) *left* (resp. *right*) *semi-i-confluence filler* for (ϕ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq \phi \odot_i \psi^{*_i}, \quad (\text{resp.} \quad |A\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq \phi^{*_i} \odot_i \psi\ ),$$

iii) *i-confluence filler* for (ϕ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq \phi^{*_i} \odot_i \psi^{*_i},$$

iv) *i-Church-Rosser filler* for (ϕ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq (\psi + \phi)^{*_i}.$$

In any n-Kleene algebra, the following inequalities hold:

$$(\psi + \phi)^{*_i} \geq \phi^{*_i} \odot_i \psi^{*_i} \geq \phi \odot_i \psi.$$

We may therefore deduce that an i-Church-Rosser filler for (ϕ, ψ) is an i-confluence filler for (ϕ, ψ) and that an i-confluence filler for (ϕ, ψ) is a local i-confluence filler for (ϕ, ψ).

9.2.2. Remarks. Conditions on the domain and codomain in the above definitions imply an i-dimensional globular character of the pair (ϕ, ψ) in the sense that we have the relation

$$|\phi^{*_i} \odot_i \psi^{*_i}\rangle_i(p) \leq |\psi^{*_i} \odot_i \phi^{*_i}\rangle_i(p)$$

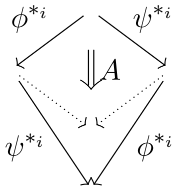
for all $p \in K_i$. Indeed, writing $A' = A \odot_j (\psi^{*_i} \odot_i \phi^{*_i})$, we have

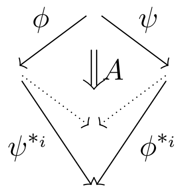
$$\begin{aligned}
|\phi^{*_i} \odot_i \psi^{*_i}\rangle_i(p) = d_i(\phi^{*_i} \odot_i \psi^{*_i} \odot_i p) &\leq d_i(d_j(A') \odot_i p) \\
&= d_i(d_j(A' \odot_i p)) \\
&= d_i(r_j(A' \odot_i p)) \\
&= d_i(r_j(A') \odot_i p) \\
&\leq d_i((\psi^{*_i} \odot_i \phi^{*_i}) \odot_i p) = |\psi^{*_i} \odot_i \phi^{*_i}\rangle_i(p),
\end{aligned}$$

where the first step holds by definition of diamonds, the second by the fact that A is an i-confluence filler and by monotonicity of d_i, the third, fourth and fifth by the globularity relations (9.1.14),(9.1.12) and (9.1.15) respectively. The final inequality follows because $d(p \cdot x) = p \cdot d(x)$ holds in modal Kleene algebra (see the end of Section 6.1.3). In the case of codomains, its dual implies that

$$r_j(A') = r_j(A \odot_j (\psi^{*_i} \odot_i \phi^{*_i})) = r_j(A) \odot_j r_j(\psi^{*_i} \odot_i \phi^{*_i}) \leq r_j(\psi^{*_i} \odot_i \phi^{*_i}).$$

The final step is again by definition of the diamond operators. Similar results hold in the case of local and semi-confluence fillers. Thus, ϕ and ψ commute modally (resp. locally modally) with respect to i-multiplication. For this reason, the confluence filler (resp. local confluence filler) defined in (9.2.1) can be represented graphically as follows





9.2.3. Whiskers. Let K be a globular modal n-Kleene algebra. Given $0 \leq i < j < n$ and $\phi \in K_j$, the *right* (resp. *left*) *i-whiskering* of an element $A \in K$ by ϕ is the element

$$A \odot_i \phi \qquad\qquad (\text{resp. } \phi \odot_i A)$$

In what follows, we list properties of whiskering and define completions.

i) Firstly, it holds that i-whiskering by j-dimensional cells semi-commutes with j-modalities. Indeed, for all $A \in K$ and $0 \leq i < j < n$ and all ϕ, ψ, γ, we have

$$\phi \odot_i \langle A|_j(\gamma) \odot_i \psi = \langle \phi \odot_i A \odot_i \psi|_j(\phi \odot_i \gamma \odot_i \psi). \tag{9.2.4}$$

Indeed, we have

$$\begin{aligned} \phi \odot_i |A\rangle_j(\gamma) &= d_j(\phi) \odot_i d_j(A \odot_j \gamma) \\ &= d_j((\phi \odot_j \phi) \odot_i (A \odot_j \gamma)) \\ &\leq d_j((\phi \odot_i A) \odot_j (\phi \odot_i \gamma)) = |\phi \odot_i A\rangle_j(\phi \odot_i \gamma), \end{aligned}$$

where the first equality is by invariance of d_j on j-dimensional elements, and by definition of j-diamonds. The second is given by globularity (9.1.14) and the inequality step by the interchange law (9.1.2) and monotonicity of domain operators. The final equality is again by definition of the j-diamond.

A strict commutation may be derived in certain cases, as explained in [17]. Indeed, let $A \in K$ be a (local) i-confluence filler for (ϕ, ψ) and $0 \leq i < j < n$. If $\phi', \psi', \gamma \in K_j$ are such that $\phi' \leq \phi$, $\psi' \leq \psi$, and $d_j(A) \leq \gamma$, we have:

$$\phi' \odot_i \langle A|_j(\gamma) \odot_i \psi' = \langle \phi' \odot_i A \odot_i \psi'|_j(\phi \odot_i \gamma \odot_i \psi). \tag{9.2.5}$$

ii) Secondly, we define *completions* of elements by whiskering. Let A be an i-confluence filler of a pair (ϕ, ψ) of elements in K_j. The *j-dimensional i-whiskering of* A is the following element of K:

$$(\phi + \psi)^{*_i} \odot_i A \odot_i (\phi + \psi)^{*_i}. \tag{9.2.6}$$

The j-star of this element is called the *i-whiskered j-completion* of A.

iii) Finally, we have that the i-whiskered j-completion of a confluence filler A, which in the following paragraph we denote by $\hat{A}$, absorbs whiskers, *i.e.* for any $\xi \leq (\phi + \psi)^{*_i}$

$$\xi \odot_i \hat{A}^{*_j} \leq \hat{A}^{*_j} \qquad \text{and} \qquad \hat{A}^{*_j} \odot_i \xi \leq \hat{A}^{*_j}. \tag{9.2.7}$$

Indeed, by definition of $\hat{A}$, we have

$$\xi \odot_i \hat{A} \leq \hat{A} \geq \hat{A} \odot_i \xi$$

for any $\xi \leq (\phi + \psi)^{*_i}$. Using the fact that $(-)^{*_j}$ is a lax morphism with respect to i-whiskering by j-dimensional elements, see Section 6.1.13, we deduce

$$\xi \odot_i \hat{A}^{*_j} \leq (\xi \odot_i \hat{A})^{*_j} \leq \hat{A}^{*_j},$$

where the last inequality holds by monotonicity of $(-)^{*_j}$. A similar proof shows that $\hat{A}^{*_j} \odot_i \xi \leq \hat{A}^{*_j}$.

9.2.8. Proposition (Coherent Church-Rosser theorem in globular n-MKA (by induction)). *Let K be a globular modal n-Kleene algebra and $0 \leq i < j < n$. Given $\phi, \psi \in K_j$, an i-confluence filler A of (ϕ, ψ) and any natural number $k \geq 0$, there exists an $A_k \leq \hat{A}^{*_j}$ such that*

- **i)** $r_j(A_k) \leq \psi^{*_i}\phi^{*_i}$,
- **ii)** $d_j(A_k) \geq (\phi + \psi)^{k_i}$,

where $\hat{A}$ is the j-dimensional i-whiskering of A.

Proof. In this proof, juxtaposition of elements denotes i-multiplication. We reason by induction on $k \geq 0$. For $k = 0$, we may take $A_0 = 1_i$. Indeed,

$$1_i \leq 1_j \leq \hat{A}^{*_j}.$$

Furthermore, we have $d_j(A_0) = 1_i = (\phi + \psi)^{0_i}$ and $r_j(A_0) = 1_i \leq \psi^{*_i}\phi^{*_i}$. Supposing that A_{k-1} is constructed, we set

$$A_k = ((\phi + \psi)A_{k-1}) \odot_j (A'\phi^{*_i}),$$

where $A' = A \odot_j (\psi^{*_i}\phi^{*_i})$. We first show that $d_j(A_k) \geq (\phi + \psi)^{k_i}$ as follows

$$\begin{aligned}
d_j(A_k) &= d_j(((\phi + \psi)A_{k-1}) \odot_j (A'\phi^{*_i})), \\
&= d_j(((\phi + \psi)A_{k-1}) \odot_j d_j(A'\phi^{*_i})), \\
&= d_j(((\phi + \psi)A_{k-1}) \odot_j d_j(A')\phi^{*_i}), \\
&\geq d_j(((\phi + \psi)A_{k-1}) \odot_j \phi^{*_i}\psi^{*_i}\phi^{*_i}), \\
&= d_j((\phi + \psi)A_{k-1}), \\
&= (\phi + \psi)d_j(A_{k-1}), \\
&= (\phi + \psi)(\phi + \psi)^{(k-1)_i}, \\
&= (\phi + \psi)^{k_i},
\end{aligned}$$

$$\begin{tikzcd}
\cdot \arrow[r, "(\phi+\psi)"] \arrow[dddrr, "\psi^{*_i}"'] & \cdot \arrow[rrr, "(\phi+\psi)^{k-1}"] \arrow[ddrr, "\psi^{*_i}"'] & & & \cdot \\
& & & (\phi+\psi)A_{k-1} & \\
& & A'\phi^{*_i} & \cdot \arrow[uur, "\phi^{*_i}"'] & \\
& & \cdot \arrow[ur, "\phi^{*_i}"'] & &
\end{tikzcd}$$

where the first step is given by definition of A_k, the second by axiom **ii)** from (6.1.3), the third by globularity (9.1.14). The inequality in the fourth step is by hypothesis that A is an i-confluence filler, and the fifth is a consequence of the fact that

$$((\phi + \psi)A_{k-1}) \odot_1 (\phi^{*_i}\psi^{*_i}\phi^{*_i}) = (\phi + \psi)A_{k-1},$$

which is in turn a consequence of the following:

$$r_j((\phi + \psi)A_{k-1}) = (\phi + \psi)r_j(A_{k-1}) \leq \phi^{*_i}\psi^{*_i}\phi^{*_i}.$$

The sixth step is again a consequence of globularity (9.1.14), the seventh follows from the induction hypothesis, and the last equality is by definition of the k-fold i-multiplication.

Now we show $r_j(A_k) \leq \psi^{*_i}\phi^{*_i}$:

$$\begin{aligned}
r_j(A_k) &= r_j(((\phi+\psi)A_{k-1}) \odot_j (A'\phi^{*_i})) \\
&= r_j(r_j((\phi+\psi)A_{k-1}) \odot_j (A'\phi^{*_i})) \\
&\leq r_j((\phi+\psi)\psi^{*_i}\phi^{*_i} \odot_j (A'\phi^{*_i})) \\
&\leq r_j((\phi^{*_i}\psi^{*_i}\phi^{*_i}) \odot_j (A'\phi^{*_i})) \\
&= r_j(d_j(A'\phi^{*_i}) \odot_j (A'\phi^{*_i})) \\
&= r_j(A')\phi^{*_i} \\
&\leq \psi^{*_i}\phi^{*_i}\phi^{*_i} \\
&= \psi^{*_i}\phi^{*_i}.
\end{aligned}$$

The first equality holds by definition of A_k, the second by axiom **ii)** from Section 6.1.3 (for codomains), the third by the induction hypothesis, the fourth by $\phi \leq \phi^{*_i}$ and $\psi\psi^{*_i} = \psi^{*_i}$. The fifth step holds since A is an i-confluence filler, the sixth by the fact that $d(x) \cdot x = x$, a consequence of axiom **i)** from Section 6.1.3. Finally, as recalled in Section 9.2.2,

$$r_j(A') = r_j(A \odot_j (\psi^{*_i} \odot_i \phi^{*_i})) = r_j(A) \odot_j r_j(\psi^{*_i} \odot_i \phi^{*_i}) \leq r_j(\psi^{*_i} \odot_i \phi^{*_i}),$$

which gives step seven since $\psi^{*_i} \odot_i \phi^{*_i} \in K_j$. The final step is due to $\phi^{*_i} \odot_i \phi^{*_i} = \phi^{*_i}$, a consequence of the Kleene star axioms.

To conclude, we must also show that $A_k \leq \hat{A}^{*_j}$. By whisker absorption, described in (9.2.3), and the fact that $A' \leq A \leq \hat{A}$, we have

$$A'\phi^{*_i} \leq \hat{A}\phi^{*_i} = \hat{A}, \qquad \text{and} \qquad (\phi+\phi)A_{k-1} \leq (\phi+\psi)\hat{A}^{*_j} \leq \hat{A}^{*_j}.$$

Thus, $A_k = ((\phi+\psi)A_{k-1}) \odot_j (A\phi^{*_i}) \leq \hat{A}^{*_j} \odot_j \hat{A}^{*_j} = \hat{A}^{*_j}$, which completes the proof. □

We now reprove this theorem using the implicit induction of Kleene algebra.

9.2.9. Theorem (Coherent Church-Rosser in globular n-MKA). *Let K be a globular n-modal Kleene algebra and $0 \leq i < j < n$. Given $\phi, \psi \in K_j$ and an i-confluence filler $A \in K$ of (ϕ, ψ), we have*

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq (\phi+\psi)^{*_i},$$

where $\hat{A}$ is the j-dimensional i-whiskering of A. Thus $\hat{A}^{*_j}$ is an i-Church-Rosser filler for (ϕ, ψ).

Proof. As in the previous proof, i-multiplication will be denoted by juxtaposition. Let ϕ, ψ be in K_j, for $0 < j < n$, and A in K be an i-confluence filler of (ϕ, ψ), with $0 \leq i < j$. By the left i-star induction axiom, see Section 6.1.13, we have

$$1_i + (\phi+\psi)|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \;\Rightarrow\; (\phi+\psi)^{*_i} \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$$

The inequality $1_i \leq \psi^{*_i}\phi^{*_i} \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$ holds. Indeed, by the unfold axiom from Section 6.1.13, we have $1_i \leq \psi^{*_i}$, $1_i \leq \phi^{*_i}$, giving the first inequality, and $1_j \leq \hat{A}^{*_j}$. The latter implies that $id_{S_{d_j}} = |1_j\rangle_j \leq |\hat{A}^{*_j}\rangle_j$, which gives $\psi^{*_i}\phi^{*_i} \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$. It then remains to show that

$$(\phi+\psi)|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}).$$

By distributivity, we may prove this for each of the summands:

- In the case of whiskering by ϕ on the left:

$$\begin{aligned}
\phi|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) &\leq |\phi\hat{A}^{*_j}\rangle_j(\phi\psi^{*_i}\phi^{*_i}) \\
&\leq |\phi\hat{A}^{*_j}\rangle_j(|A\rangle_j(\psi^{*_i}\phi^{*_i})\phi^{*_i}) \\
&\leq |\phi\hat{A}^{*_j}\rangle_j(|A\phi^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}\phi^{*_i})) \\
&\leq |\phi\hat{A}^{*_j} \odot_j A\phi^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}) \\
&\leq |\hat{A}^{*_j} \odot_j \hat{A}\rangle_j(\psi^{*_i}\phi^{*_i}) \\
&\leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})
\end{aligned}$$

$$\begin{tikzcd}
\cdot \arrow[r, "\phi"] \arrow[dddrr, "\psi^{*_i}"'] & \cdot \arrow[rrr, leftrightarrow, "(\phi+\psi)^{*_i}"] \arrow[ddrr, "\psi^{*_i}"] & & & \cdot \\
& & & \phi\hat{A}^{*_j} & \\
& & A\phi^{*_i} & \cdot \arrow[uur, "\phi^{*_i}"'] & \\
& & \cdot \arrow[ur, "\phi^{*_i}"'] & &
\end{tikzcd}$$

The first step is given by whiskering properties from (9.2.3), the second by the hypothesis that A is an i-confluence filler and that $\phi\psi^{*_i} \leq \phi^{*_i}\psi^{*_i}$. The third step is again by whiskering, and the fourth follows by definition of diamonds and axiom **ii)** from (6.1.3). The fifth follows by whisker absorption, (9.2.3), and the last step follows from the unfold axiom from (6.1.13), since it implies that $x \cdot x^* \leq x^*$.

- In the case of whiskering by ψ on the right:

$$\begin{aligned}
\psi|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) &\leq |\psi\hat{A}^{*_j}\rangle_j(\psi\psi^{*_i}\phi^{*_i}) \\
&\leq |\psi\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \\
&\leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}).
\end{aligned}$$

$$\begin{tikzcd}
\cdot \arrow[r, "\psi"] \arrow[ddrr, bend right, "\psi^{*_i}"'] & \cdot \arrow[rr, leftrightarrow, "(\phi+\psi)^{*_i}"] \arrow[ddr, "\psi^{*_i}"] & & \cdot \\
& 1_j & \psi\hat{A}^{*_j} & \\
& & \cdot \arrow[uur, "\phi^{*_i}"'] &
\end{tikzcd}$$

.

The first step is again by whiskering properties from Section 9.2.3, the second by the fact that $\psi\psi^{*_i} \leq \psi^{*_i}$ which as explained above is a consequence of the unfold axiom recalled in Section 6.1.13. Finally, whisker absorption justifies the last inequality. □

9.2.10. Remarks. Note that in Theorem 9.2.8, the elements A_k verify $|A_k\rangle_j(\psi^{*_i}\phi^{*_i}) \geq (\phi+\psi)^{k_i}$, meaning that scanning backward along A_k from $\psi^{*_i}\phi^{*_i}$, we see *at least* all of the "zig-zags" in ϕ and ψ of length k, whereas in Theorem 9.2.9, the inequality $|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq (\phi+\psi)^{*_i}$ means that scanning back from $\psi^{*_i}\phi^{*_i}$, we see *at least* all of the zig-zags in ϕ and ψ of any length. However, the elements A_k from Theorem 9.2.8 satisfy in addition

$$\langle A_k|_j((\phi+\psi)^{k_i}) \leq \psi^{*_i}\phi^{*_i}.$$

This formulation is of interest, since it coincides with the intuition of paving *from* zigzags $(\phi+\psi)^{k_i}$ *to* the confluences $\psi^{*_i}\phi^{*_i}$. However, this sort of inequality cannot be expected of the j-dimensional i-completion of A, since in general, using the path algebra intuition, $\hat{A}^{*_j}$ contains cells which go from zigzags to zigzags. In conclusion, the fact that the diamonds scan *all* possible future or past states means that we must formulate as in Theorem 9.2.9 when considering completions, or construct the paving elements as in Theorem 9.2.8.

9.2.11. Corollary. *Let K be a globular modal n-Kleene algebra. Given $\phi, \psi \in K_j$, for $i < j < n$, for any semi-i-confluence filler $A \in K$ we have*

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq (\phi+\psi)^{*_i},$$

where $\hat{A}$ is the j-dimensional i-whiskering of A.

Proof. In the case of a left semi-confluence filler, the proof is identical. If A is a right semi-confluence filler, we use the right i-star axiom and the proof is given by symmetry. □

9.3. NEWMAN'S LEMMA IN GLOBULAR MODAL n-KLEENE ALGEBRA

9.3.1. Termination in n-semirings. We define the notion of termination, or Noethericity, in a modal n-semiring K as an extension of the notion of termination in modal Kleene algebras, recalled in Section 6.2. Given $0 \leq i < j < n$, an element $\phi \in K_j$ is said to be *i-Noetherian* or *i-terminating* if

$$p \leq |\phi\rangle_i p \Rightarrow p \leq 0$$

holds for all $p \in K_i$. The set of i-Noetherian elements of K is denoted by $\mathcal{N}_i(K)$. When K is a modal p-Boolean semiring, we recall that as a consequence of the adjunction between diamonds and boxes, see Section 6.1.8, we obtain an equivalent formulation of Noethericity in terms of the forward box operator:

$$\phi \in \mathcal{N}_i(K) \qquad \Longleftrightarrow \qquad \forall p \in K_i,\ |\phi]_i p \leq p \Rightarrow 1_i \leq p.$$

We also define a notion of *well-foundedness*; ϕ is said to be i-well-founded if it is i-Noetherian in the opposite n-semiring of K.

9.3.2. Theorem (Coherent Newman's lemma for globular p-Boolean MKA). *Let K be a globular p-Boolean modal Kleene algebra, and $0 \leq i \leq p < j < n$, such that*

- **i)** $(K_i, +, 0, \odot_i, 1_i, \neg_i)$ *is a complete Boolean algebra,*

ii) *K_j is continuous with respect to i-restriction,* i.e. *for all $\psi, \psi' \in K_j$ and every family $(p_\alpha)_{\alpha \in I}$ of elements of K_i such that $sup_I(p_\alpha)$ exists, we have*

$$\psi \odot_i sup_I(p_\alpha) \odot_i \psi' = sup_I(\psi \odot_i p_\alpha \odot_i \psi').$$

Let $\psi \in K_j$ be i-Noetherian and $\phi \in K_j$ i-well-founded. If A is a local i-confluence filler for (ϕ, ψ), then

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq \phi^{*_i}\psi^{*_i},$$

i.e. *$\hat{A}^{*_j}$ is a confluence filler for (ϕ, ψ).*

Proof. We denote i-multiplication by juxtaposition. First, we define a predicate expressing restricted j-paving. Given $p \in K_i$, let

$$RP(p) \quad \Leftrightarrow \quad |\hat{A}^{*1}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq \phi^{*_i}p\psi^{*_i}.$$

By completeness of K_i, we may set $r := sup\{p \mid RP(p)\}$. By continuity of i-restriction, we may infer $RP(r)$. Furthermore, by downward closure of RP, we have the following equivalence:

$$RP(p) \quad \Longleftrightarrow \quad p \leq r.$$

This in turn allows us to make the following deductions:

$$\begin{aligned}
\forall p.\ (RP(|\phi\rangle_i p) \wedge RP(\langle\psi|_i p) \Rightarrow RP(p)) &\Leftrightarrow \forall p.\ (|\phi\rangle_i p \leq r \wedge \langle\psi|_i p \leq r \Rightarrow p \leq r) \\
&\Leftrightarrow \forall p.\ (p \leq [\phi|_i r \wedge p \leq |\psi]_i r \Rightarrow p \leq r) \\
&\Leftrightarrow [\phi|_i r \leq r \wedge |\psi]_i r \leq r
\end{aligned}$$

Thus, it suffices to show $\forall p.\ (RP(|\phi\rangle_i p) \wedge RP(\langle\psi|_i p) \Rightarrow RP(p))$ in order to conclude that $r = 1_i$, by Noethericity (resp. well-foundedness) of ψ (resp. ϕ).

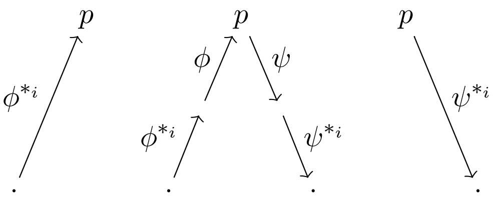
Let $p \in K_i$, set $|\phi\rangle_i(p) = p_\phi$ and $\langle\psi|_i(p) = p_\psi$ and suppose that $RP(p_\phi)$ and $RP(p_\psi)$ hold. Note that we have

$$\phi p = d_i(\phi p)\phi p = |\phi\rangle_i(p)\phi p \leq p_\phi \phi,$$

since $d(x)x = x$ by axiom **i)** from Section 6.1.3 and $p \leq 1_i$. We have a similar inequality for ψ, that is $p\psi \leq \psi p_\psi$. These inequalities, along with the unfold axioms from Section 6.1.13,

give $\phi^{*_i}p\psi^{*_i} \leq \phi^{*_i}p + \phi^{*_i}\phi p\psi\psi^{*_i} + p\psi^{*_i}$

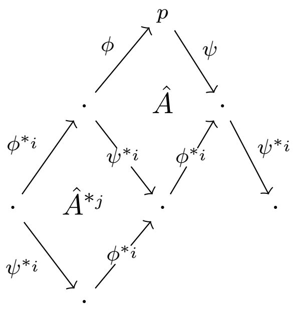
$$\leq \phi^{*_i}p + \phi^{*_i}p_\phi\phi\psi p_\psi\psi^{*_i} + p\psi^{*_i}.$$



The outermost summands are below $|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$. Indeed, $id_{S_j} = |1_j\rangle_j \leq |\hat{A}^{*_j}\rangle_j$ since $1_j \leq \hat{A}^{*_j}$, $p \leq 1_i$ and $\phi^{*_i}, \psi^{*_i} \leq \psi^{*_i}\phi^{*_i}$.

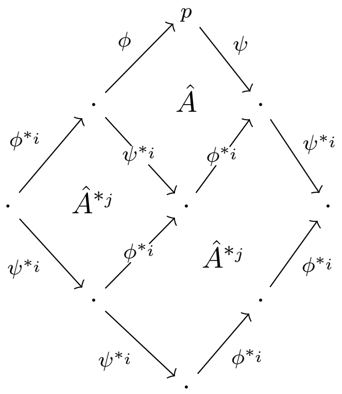
For the middle summand, we calculate

$$\begin{aligned}
\phi^{*_i} p_\phi \phi\psi p_\psi \psi^{*_i} &\le \phi^{*_i} p_\phi |A\rangle_j (\psi^{*_i}\phi^{*_i}) p_\psi \psi^{*_i}\\
&\le |\phi^{*_i} p_\phi A p_\psi \psi^{*_i}\rangle_j (\phi^{*_i} p_\phi \psi^{*_i}\phi^{*_i} p_\psi\psi^{*_i})\\
&\le |\phi^{*_i} p_\phi \hat{A} p_\psi \psi^{*_i}\rangle (|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})\phi^{*_i}p_\psi\psi^{*_i})\\
&\le |\hat{A}\rangle(|\hat{A}^{*_j}\phi^{*_i}p_\psi\psi^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}p_\psi\psi^{*_i}))\\
&\le |\hat{A}\odot_j \hat{A}^{*_j}\phi^{*_i}p_\psi\psi^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}p_\psi\psi^{*_i})\\
&\le |\hat{A}\odot_j\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}p_\psi\psi^{*_i}),
\end{aligned}$$



The first step is by the local i-confluence filler hypothesis, the second by whiskering properties from Section 9.2.3 and the third by $RP(p_\phi)$. The fourth step is again by whiskering properties, and the fifth follows from axiom **ii)** in Section 6.1.3 and the definition of diamond operators. The final step is by whisker absoprtion, see Section 9.2.3. By similar arguments, we have

$$\begin{aligned}
|\hat{A}\odot_j\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}p_\psi\psi^{*_i}) &\le |\hat{A}\odot_j\hat{A}^{*_j}\rangle_j(\psi^{*_i}|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}))\\
&\le |\hat{A}\odot_j\hat{A}^{*_j}\rangle_j(|\psi^{*_i}\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}))\\
&\le |\hat{A}\odot_j\hat{A}^{*_j}\odot_j\psi^{*_i}\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})\\
&\le |\hat{A}\odot_j\hat{A}^{*_j}\odot_j\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})
\end{aligned}$$



Indeed, the first step follows from $RP(p_\psi)$, and the second by whiskering properties. The third step follows from axiom **ii)** in Section 6.1.3 and the definition of diamond operators as in the preceding calculation. The final step follows from whisker absorption. Finally, we observe that

$$\hat{A}\odot_j\hat{A}^{*_j}\odot_j\hat{A}^{*_j}\le\hat{A}^{*_j},$$

and thus by monotonicity of the diamond operator we may conclude that

$$\phi^{*_i}p_\phi\phi\psi p_\psi\psi^{*_i}\le|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}).$$

We have thereby shown that $\forall p(RP(p_\phi)\wedge RP(p_\psi)\Rightarrow RP(p))$ and thus that $r=1_i$, concluding the proof. □

9.3.3. Remark. Similarly to the discussion from Remark 7.1.18 in the context of polygraphs, we remark here that the proofs of Theorems 9.2.9 and 9.3.2 are similar to those of the analogous 1-dimensional results for modal Kleene algebra found in [28, 109]. Indeed, if we look exclusively at the induction axioms and deductions applied to j-dimensional cells, we obtain the same proof structures as in the case of modal Kleene algebras. This indicates that the structure of globular modal n-Kleene algebra is a natural higher dimensional generalisation of modal Kleene algebras in which proofs of coherent confluence may be calculated. The consistency of the abstract, algebraic results from the previous sections with the point-wise, polygraphic results from Section 8.1.1 are made explicit in Section 9.5.

9.4. ABSTRACT COHERENCE IN HIGHER KLEENE ALGEBRAS

Just as in Section 7.3.4 when we looked at the abstract coherence theorem for n-polygraphs, we will see in this section that the abstract coherence theorem for higher Kleene algebra is a consequence of the two dimensional case. We fix a p-Boolean n-Kleene algebra K.

9.4.1. Sections, skeleta and strategies in HKA. In Section 8.4.1, we defined sections, skeleta and strategies in the context of Boolean MKA. When a n-Kleene algebra is p-Boolean, we know that each K_j, for $j \leq p$, is a Boolean MKA with respect to $(j-1)$-operations. For $i \leq j$, we therefore also have Boolean complementation on K_i by restriction. Let $x \in K_j$.

- **i)** The *i-equivalence* generated by x is the element $x^{\top_i} := (x + \overline{x})^{*_i}$. For $p \in K_i$, the *x-saturation* of p is the element $|x^{\top_i}\rangle_i(p) \in K_i$.
- **ii)** An *i-covering set* for x is an element $q \in K_i$ such that $|x^{\top_i}\rangle_i(q) \geq 1_i$, *i.e.* whose x-saturation is total. An *i-section* of x is a minimal i-covering set.
- **iii)** A *wide i-sub* of x is an element$w \in K_j$ such that $w \leq x$ and $|w\rangle_i = |x\rangle_i$ and $\langle w|_i = \langle x|_i$. A *$i$-skeleton* of x is a minimal wide sub.
- **iv)** Given an i-section s of x, an *i-strategy for x relative to s* is an i-skeleton σ of $x^{\top_i} \odot_i s$ such that $s \odot_i \sigma \leq s$.

9.4.2. Theorem (Coherent normalising Newman's lemma [16]). *Let $i < j < n$ and p such that $i \leq p$. Let K be a p-Boolean globular n-Kleene algebra such that*

- **i)** *$(K_i, +, 0, \odot_i, 1_i, \neg_i)$ is a complete Boolean algebra,*
- **ii)** *K_j is continuous with respect to i-restriction, that is for all $\psi, \psi' \in K_j$ and $(p_\alpha)_\alpha \subseteq K_i$ we have $\psi \odot_i \vee p_\alpha \odot_i \psi' = \vee(\psi \odot_i p_\alpha \odot_i \psi')$.*

Let $\phi \in K_j$ be convergent and σ be a skeleton of $exh(\phi)_j$. If A is a local confluence filler for $(\overline{\phi}, \phi)$, then $|\hat{A}^{*_j}\rangle_j(\sigma \odot_i \overline{\sigma}) \geq \overline{\phi}^{*_i} \odot_i \phi^{*_i}$.

Proof. The proof is identical to that of Theorem 8.5.1 in which 0-operations (resp. 1-operations) are replaced by i-operations (resp. j-operations). This is because

$$(K, +, 0, \odot_i, 1_i, d_i, r_i, \neg_i, \odot_j, 1_j, d_j, r_j)$$

is a 0-Boolean globular 2-Kleene algebra. □

9.4.3. Theorem (Abstract coherence theorem for HKA [16]). *Let K be a p-Boolean globular n-Kleene algebra satisfying the additional hypotheses in Theorem 9.4.2 and $\phi \in K_j$ convergent. Given a normalisation strategy σ for ϕ and a local confluence filler A for $(\overline{\phi}, \phi)$, we have*

$$|\hat{A}^{*_j}\rangle_j(\sigma \odot_i \overline{\sigma}) \geq \phi^{\top_i} = (\phi + \overline{\phi})^{*_i}.$$

Proof. As above, the proof is identical to that of Theorem 8.5.2 in which 0-operations (resp. 1-operations) are replaced by i-operations (resp. j-operations). □

9.5. POLYGRAPHIC COHERENCE VIA HKA

In this section we interpret the theorems from the preceding section in terms of polygraphs. We fix an n-polygraph P and a cellular extension Γ of $P_n^\top$.

9.5.1. Converses. Recall from Section 6.1.17 that a Kleene algebra with converse is a Kleene algebra K equipped with an involution $\overline{(-)} : K \to K$ that distributes through addition, acts contravariantly on multiplication, commutes with the Kleene star, *i.e.*

$$\overline{(a+b)} = \overline{a} + \overline{b}, \qquad \overline{(a \cdot b)} = \overline{b} \cdot \overline{a}, \tag{9.5.1}$$

$$\overline{(a^*)} = (\overline{a})^*, \qquad \overline{(\overline{a})} = a, \tag{9.5.2}$$

and satisfies the inequality $a \leq a\overline{a}a$, in the case of a Gelfand converse, or the inequality $d(x) \leq x\overline{x}$ in the case of a contracting converse. When the underlying Kleene algebra is a modal Kleene algebra, we say that it is a modal Kleene algebra with converse.

9.5.2. (n, p)-Kleene algebra. A modal *Gelfand* (resp. *contracting*) (n, p)-*Kleene algebra* K is a modal n-Kleene algebra equipped with operations $\overline{(-)}^j : K_{j+1} \to K_{j+1}$ for $p \leq j < n-1$ and an operation $\overline{(-)}^{n-1} : K \to K$, satisfying the axioms listed in (9.5.1) and the Gelfand inequality (6.1.19) (resp. the contraction inequality (6.1.20)) relative to the appropriate multiplication operation, *i.e.* $\overline{(-)}^j$ is a converse for the j-multiplication. Note that for $\phi \in K_i$ with $i < j$, we have $\overline{\phi}^j = \phi$. This is a consequence of the fact that for $i \leq j$, $\odot_j$ is idempotent for elements of K_i.

9.5.3. Conversion in $K(P,\Gamma)$. The modal $(n+1)$-Kleene algebra $K(P,\Gamma)$, as defined in Section 9.1.19, is a modal $(n+1, n-1)$-Kleene algebra. Indeed, for any $\phi \in K(P,\Gamma)_n$ and $A \in K$, let

$$\overline{\phi}^{n-1} := \{ u^- \mid u \in \phi \} \qquad \text{and} \qquad \overline{A}^{n} = \{ \alpha^- \mid \alpha \in A \}.$$

This operation is well defined in the following sense: If $\phi \in K(P,\Gamma)_n$, then ϕ is a set of cells of dimension less than or equal to n. Given a cell v of dimension $i < n$, its n-inverse is itself, since we consider it as an identity. Given a cell u of dimension n, we know that u^- is well defined since if $u \in P_n^\top$ then $u^- \in P_n^\top$. Similarly for the case of $\overline{(-)}^{n}$.

9.5.4. Γ-coherence properties as fillers. Recall that Γ and P_n^* are themselves elements of $K(P,\Gamma)$, and that in Proposition 9.1.20 we observed that

$$\Gamma^c = (1_n \odot_{n-1} (\cdots \odot_2 (1_2 \odot_1 (1_1 \odot_0 \Gamma \odot_0 1_1) \odot_1 1_2) \odot_2 \cdots) \odot_{n-1} 1_n),$$

where Γ^c is the set of cells of Γ in context. In the following, we will denote by P_n^c the set of rewriting steps generated by P_n, which can be expressed in $K(P,\Gamma)$ as

$$P_n^c = (1_{n-1} \odot_{n-2} (\cdots \odot_2 (1_2 \odot_1 (1_1 \odot_0 P_n \odot_0 1_1) \odot_1 1_2) \odot_2 \cdots) \odot_{n-2} 1_{n-1}).$$

The construction of $K(P,\Gamma)$ is compatible with Γ-coherence properties in the following sense:

9.5.5. Proposition. *With $\Gamma' := (\Gamma^c)^{*_n}$, the following equivalences hold:*

- **i)** *Γ is a (local) confluence filler for P $\iff$ Γ' is a (local) $(n-1)$-confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$,*
- **ii)** *Γ is a Church-Rosser filler for P $\iff$ Γ' is an $(n-1)$-Church-Rosser filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$.*

Proof. Let us prove the equivalence in the case of (global) confluence.

Suppose that Γ is a confluence filler for P. An element $f^- \star_{n-1} g \in \overline{(P_n^c)}^{n-1} \odot_{n-1} P_n^c$ corresponds to a branching (f,g). By hypothesis, there exists an $\alpha \in P_n^\top[\Gamma]$ such that $s_n(\alpha) = f^- \star_{n-1} g$ and α is an n-composition of rewriting steps so $\alpha \in \Gamma'$. Furthermore, the n-target of α is a confluence, so $\alpha \in \Gamma' \odot_n (P_n^c \odot_{n-1} \overline{(P_n^c)}^{n-1})$. In equations, this means that

$$\overline{(P_n^c)}^{n-1} \odot_{n-1} P_n^c \subseteq d_n \left(\Gamma' \odot_n (P_n^c \odot_{n-1} \overline{(P_n^c)}^{n-1}) \right) = |\Gamma'\rangle_n \left(P_n^c \odot_{n-1} \overline{(P_n^c)}^{n-1} \right),$$

i.e. Γ' is an $(n-1)$-confluence filler for $\left(\overline{(P_n^c)}^{n-1}, P_n^c\right)$.

Conversely, if Γ' is an $(n-1)$-confluence filler for $\left(\overline{(P_n^c)}^{n-1}, P_n^c\right)$, then given some branching (f,g), we know that $f^- \star_{n-1} g \in d_i \Gamma' \odot_n (P_n^c \odot_{n-1} \overline{(P_n^c)}^{n-1})$. This means there exists some cell $\alpha \in \Gamma'$ with n-source $f^- \star_{n-1} g$ and whose n-target is a confluence.

Since $\alpha \in \Gamma'$, we know that it is a composition of rewriting steps of Γ. With this we conclude that P is Γ-confluent.

The other cases are similarly deduced. $\square$

Due to this compatibility, we may deduce the following theorems, that is Theorems 8.1.4 and 8.1.7, as corollaries of our main results:

9.5.6. Theorem (Church Rosser for n-polygraphs). *Let P be an n-polygraph and Γ a cellular extension of $P_n^\top$. Then Γ is a confluence filler for P if, and only if, Γ is a Church-Rosser filler for P.*

Proof. Suppose first that Γ is a confluence filler for P. Using the result and notations from Proposition 9.5.5, we know that Γ' is an $(n-1)$-confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. We apply Theorem 9.2.9 to $K(P,\Gamma)$ for $i = n-1$ and $j = n$, obtaining that $\widehat{\Gamma'}^{*n}$ is an $(n-1)$-Church-Rosser filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. Observing that $(P_n^c + \overline{(P_n^c)}^{n-1})^{*_{n-1}} = P_n^\top$, we have

$$\widehat{\Gamma'}^{*n} = \Big(P_n^\top \odot_{n-1} (\Gamma^c)^{*_n} \ \ \odot_{n-1} P_n^\top\Big)^{*n} \subseteq \Big((P_n^\top \odot_{n-1} \Gamma^c \ \ \odot_{n-1} P_n^\top)^{*_n}\Big)^{*n} = \Gamma',$$

where the first step is by definition, the second uses the fact that the n-star is a lax morphism for $(n-1)$-multiplication, see Section 9.1.16, and the third uses the fact that Γ^c absorbs whiskers and that $(A^{*_n})^{*_n} = A^{*_n}$. Since additionally, $\Gamma' \subseteq \widehat{\Gamma'}^{*n}$, Γ' is an $(n-1)$-Church-Rosser filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. By Proposition 9.5.5, this allows us to conclude that Γ is a Church-Rosser filler for P.

For the trivial direction, suppose that Γ is a Church-Rosser filler for P. We deduce by Proposition 9.5.5 that Γ' is an $(n-1)$-Church-Rosser filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. As pointed out at the end of Section 9.2.1, this means that Γ' is an i-confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$, by which we conclude that Γ is a confluence filler for P. $\square$

9.5.7. Theorem (Newman for n-polygraphs). *Let P be a terminating n-polygraph and Γ a cellular extension of $P_n^\top$. Then Γ is a local confluence filler for P if, and only if, Γ is a confluence filler for P.*

Proof. Suppose that Γ is a local confluence filler for P. Using the result and notations from Proposition 9.5.5, we know that Γ' is an $(n-1)$-local confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. We apply Theorem 9.3.2 to $K(P,\Gamma)$ for $i = n-1$ and $j = n$, obtaining that $\widehat{\Gamma'}^{*n}$ is an $(n-1)$-confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. As in the proof of the previous theorem, we have that $\widehat{\Gamma'}^{*n} = \Gamma'$, allowing us to conclude that Γ is a confluence filler for P, again by Proposition 9.5.5.

For the trivial direction, suppose that Γ is a confluence filler for P. As above, we deduce that Γ' is an $(n-1)$-Church-Rosser filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$. Again, as pointed out at in Section 9.2.1, this means that Γ' is a local i-confluence filler for $(\overline{(P_n^c)}^{n-1}, P_n^c)$, by which we conclude that Γ is a local confluence filler for P via Proposition 9.5.5. $\square$

CHAPTER 10.
A JÓNSSON-TARSKI DUALITY FOR HIGHER STRUCTURES

In this chapter, we briefly recount a work in progress between the author, P. Malbos, D. Pous and G. Struth [18]. The work in question centres around the formalisation of rewriting techniques in higher settings, as well as in convolution algebras, and can be seen as a continuation of [17] and [16] on the one hand, and [41] and the later publication including the author [14]. That is, the Kleene algebraic structures recalled in Chapter 9 from [17] and their applications to coherence problems [16] described in Chapter 8 and Section 9.4, and in particular the lifting of free higher categories to HKA described in Section 9.1.19, were discovered to be related to the notion of ℓr-multisemigroups [41], now called *catoids*, and structures called *quantales*, see [42] or the standard reference [101]. This sparked the collaboration [14], which has now led to the exploration of Jonssón-Tarski style correspondence theorems relating catoids to quantales in a higher dimensional setting.

In Sections 10.1 and 10.2 we recall the notions of catoid and of modal quantale from [41, 42], respectively. Then, in Sections 10.4 and 10.3 we introduce their higher dimensional counterparts [18]. Finally, in Section 10.5, we state Theorems 10.5.1 and 10.5.2, the main theorems of this chapter, which express a correspondence between HKA and higher quantales.

We do not present proofs of the results found in this section as they have all been formalised in the proof assistant Isabelle, see G. Struth's github repository, which contains the author's contributions in the formalisation of higher quantales.

10.1. CATOIDS

We start by recalling definitions of catoids and related structures [14, 41]. Catoids were previously called *ℓr-multisemigroups*, but this name quickly becomes unwieldy when considering specifics, such as locality or functionality. General background on multisemigroups can be found in [81].

10.1.1. Catoids. Catoids generalise 1-categories, providing a single-set approach to categorical structures in which the composition operation is a multi-operation. A *(small) catoid* is a structure $(X, \odot, \ell, r)$ consisting of a set X, a multioperation $\odot : X \times X \to \mathcal{P}X$ and *source* and *target* maps $\ell, r : X \to X$ that satisfy, for all $x, y, z \in X$,

$$\bigcup\{x \odot v \mid v \in y \odot z\} = \bigcup\{u \odot z \mid u \in x \odot y\},$$
$$x \odot y \neq \emptyset \Rightarrow r(x) = \ell(y), \qquad \ell(x) \odot x = \{x\}, \qquad x \odot r(x) = \{x\}.$$

A catoid X is *functional* if $x, x' \in y \odot z$ imply $x = x'$ for all $y, z \in X$ and *local* if $r(x) = \ell(y) \Rightarrow x \odot y \neq \emptyset$ for all $x, y \in X$. A *single-set category* is a local functional catoid.

The partiality of the composition operation $\odot$ of a functional catoid is encoded via the empty set, *i.e.* $x \odot y = \emptyset$ when x and y are not composable. We say that the composition of x and y is *defined* if $x \odot y \neq \emptyset$. When this is the case, we write $\Delta(x, y)$. Following this logic, we say that the operation is *total* if for all $x, y \in X$ there is exactly one $z \in X$ such that $z \in x \odot y$. In this case, the structure is isomorphic to a monoid.

The first catoid axiom expresses associativity of $\odot$. To see this more easily, we lift $\odot$ to an operation $\odot : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$ defined, for all $A, B \subseteq X$, as

$$A \odot B = \bigcup_{x \in A, y \in B} x \odot y,$$

and denote $x \odot A$ as $\{x\} \odot A$. The first axiom then becomes, for all $x, y, z \in X$,

$$x \odot (y \odot z) = (x \odot y) \odot z.$$

Structures equipped with an associative multioperation are known as multisemigroups, see [81]. The second catoid axiom states that if the composition of x and y is defined, then the target $r(x)$ of x equals the source $\ell(y)$ of y, expressing the relationship between composition on the one hand and source and target on the other. This captures the composition pattern found in 1-categories, since in local dioids, $\Delta(x, y) \Leftrightarrow r(x) = \ell(y)$. The third and fourth catoid axioms express other relations between composition and the source and target maps, in that they express that $\ell(x)$ is a left unit and $r(x)$ a right unit of x. We refer to these axioms as *absorption* axioms.

On one hand, catoids therefore generalise single-set categories beyond locality and functionality. On the other hand they correspond to multimonoids seen as multisemigroups with many units. As explained above, in the multioperational setting, partiality of composition is captured by mapping to the empty set. For a total operation, therefore, $\Delta = X \times X$. From now on, we will write $xy := x \odot y$ if no confusion is possible.

10.1.2. The category of catoids. A *catoid morphism* $f : X \to Y$ between catoids X and Y satisfies $f(x \odot_X y) \subseteq f(x) \odot_Y f(y)$ and preserves ℓ and r: $f \circ \ell_X = \ell_Y \circ f$ and $f \circ r_X = r_Y \circ f$. Catoids and their morphisms form a category.

A morphism $f : X \to Y$ is *bounded* if $f(x) \in u \odot_Y v$ implies that there are $y, z \in X$ such that $x \in y \odot_X z$, $u = f(y)$ and $v = f(z)$. Bounded morphisms appear in modal and substructural logics; they correspond to functional bisimulations [114].

10.1.3. Example. Bounded morphisms need not satisfy $f(x \odot_X y) = f(x) \odot_Y f(y)$. Consider the discrete category on $X = \{a, b\}$ as a catoid. It has source and target maps $\ell = id_X = r$ and thus composition

$$a \odot b = \begin{cases} \{a\} & \text{if } a = b, \\ \emptyset & \text{if } a \neq b. \end{cases}$$

The constant map $f_b : x \mapsto b$ on X is clearly a catoid morphism. It is bounded because every $x \in X$ satisfies $f_b(x) \in b \odot b$, $x \in x \odot x$ and $b = f_b(x)$. Nevertheless $f_b(a \odot b) = \emptyset \neq \{b\} = f_b(a) \odot f_b(b)$.

Many examples of catoids and related structures, from mathematics and computing, can be found in [41] and [14]. Here we recall how 1-categories, and in particular free 1-categories, may be captured in this setting.

10.1.4. Catoids and categories. We have already defined (small) single-set categories as local functional catoids. This is justified by the fact that local functional catoids with (bounded) morphisms and single-set categories à la Mac Lane with the same morphisms are isomorphic as categories [41]. The elements of single-set categories are 1-cells of small categories. The 0-cells of the latter correspond bijectively to identity 1-cells and thus to units of catoids. Morphisms of local functional catoids correspond to functors of categories.

In the notation of Section 5.2.1, given a category $\mathcal{C}$, we construct $(X_{\mathcal{C}}, \ell, r)$, where the underlying set $X_{\mathcal{C}}$ is given by $\mathcal{C}_1$, the set of 1-cells of $\mathcal{C}$. Note that this set contains the identity morphism 1_c associated to each object $c \in \mathcal{C}_0$. The composition operation is given by $f \odot g = \{f \odot g\}$, and the source and target maps ℓ and r are provided by the maps s_0 and t_0 respectively.

10.1.5. Free categories. The free category P^* over a 1-polygraph $\ell, r : P_1 \to P_0$, see Section 5.2.3, for a set P_1 of generating 1-cells and a set P_0 of 0-cells can be defined in single-set-style as follows. We consider (X_P, ℓ, r) satisfying $\ell \circ \ell = \ell$, $r \circ r = r$, $\ell \circ r = r$ and $r \circ \ell = \ell$. These conditions mean that $X_\ell = X_r$ corresponds to the set of 0-cells in the guise of identity 1-cells, while its complement $X \setminus X_\ell$ corresponds to the set of (non-identity) 1-cells. Compare this with the compatibility conditions for domain and codomain operations in modal Kleene algebras, see Section 6.1.5.

10.1.6. Remark. Multioperations $X \times X \to \mathcal{P}X$ are isomorphic to ternary relations on X (as maps $X \to X \to X \to 2$). A *relational monoid* is then a monoid in the monoidal category of relations with the standard tensor. It has many units and can alternatively be seen as a relational semigroup equipped with an ℓr-structure [24]. More concretely, a catoid can then be defined as a relational structure (X, R, ℓ, r) which satisfies the associativity law $\exists v.\ R^w_{xv} \wedge R^v_{yz} \Leftrightarrow \exists u.\ R^u_{xy} \wedge R^w_{uz}$, the definedness law $\exists z.\ R^z_{xy} \Rightarrow r(x) = \ell(z)$ and the absorption laws $R^x_{\ell(x)x}$ and $R^x_{xr(x)}$, where we write R^x_{yz} instead of $(x, y, z) \in R$. Properties of functionality or locality can be translated into the relational setting along the same lines.

Ternary and more generally $(n + 1)$-ary relations appear as duals of binary and more generally n-ary modal operators in Jónsson and Tarski's duality theory for Boolean algebras with operators [49, 50, 78, 79]. While catoids based on multirelations support intuitive algebraic and set-theoretic reasoning, their relational siblings are more suitable for type-theoretic approaches with functions $X \to X \to X \to 2$ representing ternary relations.

10.2. Modal Quantales

Now we recall the definition of modal quantales from [42]. Quantales [101] are complete lattices with a continuous multiplication operation which constitute a specific case of Kleene algebras. We augment these structures with modal operators defined via domain and codomain operations in the style of MKA, see Section 6.1. The Isabelle theory concerning quantalic structures can be found here.

Recall that a *quantale* $(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ with a monoidal composition $\cdot$ and unit 1 that preserves all sups in both arguments, *i.e.* is continuous. We write $\bigvee$, $\vee$, $\bigwedge$ and $\wedge$ for sups, binary sups, infs and binary infs in a quantale, and $\bot$ for the smallest and $\top$ for the greatest element. A *subidentity* of Q is an element $x \leq 1$.

A quantale is *distributive* if its underlying lattice satisfies $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and thus its dual property $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It is *Boolean* if its underlying lattice is a Boolean algebra. We denote Boolean complementation by $-$ to distinguish it from the case of Kleene algebras, see Section 6.1.

Due to the continuity of multiplication with respect to multiplication, quantales are in fact Kleene algebras. Indeed, a Kleene star $(-)^* : Q \to Q$ can be defined on any quantale Q, for $x^0 = 1$ and $x^{i+1} = xx^i$, as

$$x^* = \bigvee_{i \geq 0} x^i.$$

A *domain quantale* [42] is a quantale Q with an operation $d : Q \to Q$ such that, for all $x, y \in Q$,

$$x \leq d(x)x, \qquad d(xd(y)) = d(xy), \qquad d(x) \leq 1,$$
$$d(\bot) = \bot, \qquad d(x \vee y) = d(x) \vee d(y).$$

These domain axioms are the same as those for domain semirings [29], see Section 6.1.3. As for catoids, we refer to the first axiom as the *absorption axiom*. The second axiom expresses *locality* of d. The third axiom is the *subidentity* axiom, the fourth the *bottom* axiom and the final the *(binary) sup* axiom. Most properties of interest translate from domain semirings to domain quantales. In addition, d preserves all sups and all non-empty infs [42].

Similarly to catoids and domain semirings, $Q_d = \{x \in Q \mid d(x) = x\} = d(Q)$. It follows that $(Q_d, \leq, \cdot, 1)$ is a subquantale of Q that forms a bounded distributive lattice in which $\cdot$

equals $\wedge$. We call Q_d the lattice of *domain elements* or simply the *domain algebra*. In a Boolean quantale, Q_d is the set of all subidentities.

Quantales are closed under opposition, which exchanges the arguments in compositions. A *codomain quantale* (Q, r) is then a domain quantale (Q^{op}, d). A *modal quantale* is a domain and codomain quantale $(Q, \leq, \cdot, 1, d, r)$ that satisfies the compatibility laws

$$d \circ r = r \qquad \text{and} \qquad r \circ d = d.$$

These guarantee $Q_d = Q_r$. In the case of Boolean quantales, these compatibility axioms are not necessary. Again, we invite the reader to recall similarities with modal semirings, see Section 6.1.

10.2.1. Remark. Locality in the form $x \odot y \neq \emptyset \Leftrightarrow r(x)\ell(y) \neq \emptyset$ corresponds to

$$x \cdot y \neq \bot \Leftrightarrow r(x) \cdot d(y) \neq \bot$$

in modal quantales, which is a consequence of locality of d and r in modal semirings, and hence in modal quantales. In modal power-set quantales, it is even equivalent to locality of d and r. Yet the more precise properties $r(x)\ell(y) \neq \emptyset \Leftrightarrow r(x) = \ell(y)$ and $\Delta(x, y) \Leftrightarrow r(x) = \ell(y)$ do not lift to modal power-set quantales. Let A, for instance, consist of a path π_1 with target v and a path π_2 with target v, and let B consist of a path π_3 with source v. Then $r(A) = \{v, w\} \neq \{v\} = d(B)$, but $A \cdot B$ consists of the path obtained from gluing π_1 and π_2 at their ends ,see [89] for details on such path algebras, or revisit the path model presented in Example 6.1.22. In turn,

$$r(\{a\}) \cap d(\{b\}) \neq \emptyset \Leftrightarrow \{r(a)\} \cap \{\ell(b)\} \neq \emptyset \Leftrightarrow \{r(a)\} = \{\ell(b)\} \Leftrightarrow r(\{a\}) = d(\{b\})$$

holds at least in the atom structure of any modal power set quantale.

10.3. GLOBULAR n-CATOIDS AND SINGLE-SET n-CATEGORIES

MacLane has generalised single-set categories to single-set 2-categories in Chapter XII of his book [89], and pointed out that a single-set strict n-category is obtained by imposing a 2-category structure on every pair (i, j) of single-set categories X_i and X_j for $0 \leq i < j \leq n$, see Section 5.4.1. This can be generalised to catoids, thereby obtaining higher catoids.

A *(globular) n-catoid* is a structure $(X, \odot_i, \ell_i, r_i)_{0 \leq i < n}$ such that each $(X, \odot_i, \ell_i, r_i)$ is a catoid and the structures interact, for all $0 \leq i < j < n$, via the following axioms:

$$\ell_i \circ \ell_j = \ell_j \circ \ell_i, \qquad \ell_i \circ r_j = r_j \circ \ell_i, \qquad r_i \circ \ell_j = \ell_j \circ r_i, \qquad r_i \circ r_j = r_j \circ r_i,$$

$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z),$$

$$\ell_j(x \odot_i y) = \ell_j(x) \odot_i \ell_j(y), \qquad r_j(x \odot_i y) = r_j(x) \odot_i r_j(y),$$

$$\ell_i(x \odot_j y) \subseteq \ell_i(x) \odot_j \ell_i(y), \qquad r_i(x \odot_j y) \subseteq r_i(x) \odot_j r_i(y),$$

$$\ell_j \circ \ell_i = \ell_i, \qquad \ell_j \circ r_i = r_i, \qquad r_j \circ \ell_i = \ell_i, \qquad r_j \circ r_i = r_i.$$

A *(single-set) n-category* is a local functional n-catoid, that is, each $(X, \cdot_i, \ell_i, r_i)$ is local and functional.

Intuitively, the elements of X describe (generalised) n-cells, while the sets X_{ℓ_i}, for $i < n$, describe (generalised) i-cells as degenerate n-cells of lower dimension. This is similar to, but not precisely the same as, the model of n-Kleene algebra given by the power-set lifting of the free category generated by an n-polygraph, see Section 9.1.19. This is made clear in Section 10.5. In fact, the atoms of $K(P, \Gamma)$, see Section 9.1.19, have the structure of a single-set n-category.

The identities in the last row of the above definition impose that all identity arrows of the 0-structure are also identities of the 1-structure. The homomorphism laws in its third row state that horizontal compositions of vertical identity arrows are vertical identity arrows of the composite cells.

$$\begin{tikzcd}
\ell_0(x) \arrow[r, bend left, "\ell_1(x)"] \arrow[r, bend right, "r_1(x)"'] \arrow[r, phantom, "\Downarrow x"] & r_0(x) \arrow[r, bend left, "\ell_1(y)"] \arrow[r, bend right, "r_1(y)"'] \arrow[r, phantom, "\Downarrow y"] & r_0(y)
\end{tikzcd}$$

This explains the use of equality in these laws.

The interchange law, on the other hand, is not equational because the right-hand diagram below is defined on the right of the interchange law, but not on the left, while the left-hand diagram is defined on both sides.

$$\begin{tikzcd}
\ell_0(w) \arrow[r] \arrow[r, bend left=50, "\ell_1(w)"] \arrow[r, bend right=50, "r_1(x)"'] \arrow[r, phantom, shift left=2.5, "\Downarrow w"] \arrow[r, phantom, shift right=2.5, "\Downarrow x"] & r_0(x) \arrow[r] \arrow[r, bend left=50, "\ell_1(y)"] \arrow[r, bend right=50, "r_1(z)"'] \arrow[r, phantom, shift left=2.5, "\Downarrow y"] \arrow[r, phantom, shift right=2.5, "\Downarrow z"] & r_0(z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\ell_0(x) \arrow[r] \arrow[r, bend right=50, "r_1(x)"'] \arrow[r, phantom, shift right=2.5, "\Downarrow x"] \arrow[rr, bend left=40, "\ell_1(w)"] \arrow[rr, phantom, shift left=4, "\Downarrow w"] & \ell_0(x) \arrow[r] \arrow[rr, bend right=40, "r_1(z)"'] \arrow[rr, phantom, shift right=4, "\Downarrow z"] & \ell_0(x) \arrow[r] \arrow[r, bend left=50, "\ell_1(y)"] \arrow[r, phantom, shift left=2.5, "\Downarrow y"] & \ell_0(x)
\end{tikzcd}$$

This is usually not made explicit when defining higher categorical structures, although Steiner makes it clear in [105]. Recall the similarity between the above and the lax interchange law defined in the setting of higher Kleene algebras in Section 9.1. The above shows that the interchange law of higher categories, usually expressed as an equality, is in fact an inclusion when the partiality of composition is modelled by mapping pairs to the empty set. Furthermore, we see that the lax interchange law for HKA is expressing the same phenomenon in the case of sets of higher cells; this is made clear in the proof of Proposition 9.1.20.

Likewise, the homomorphism laws in the third row are inclusions because the left-hand diagram below is defined on the right of the first homomorphism law, but not on the left, and the right-hand diagram on the right of the second homomorphism law, but not on the left, while the diagram in the middle is defined on both sides of each.

$$\begin{tikzcd}
& r_0(x) \\
\ell_0(x) \arrow[ur, bend left, "\ell_1(x)"] \arrow[ur, bend right, "r_1(x)"'] \arrow[ur, phantom, "\Rightarrow x"] \arrow[dr, bend left, "\ell_1(y)"] \arrow[dr, bend right, "r_1(y)"'] \arrow[dr, phantom, "\Leftarrow y"] & \\
& r_0(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\ell_0(x) \arrow[r, bend left=50, "\ell_1(x)"] \arrow[r] \arrow[r, bend right=50, "r_1(y)"'] \arrow[r, phantom, shift left=2, "\Downarrow x"] \arrow[r, phantom, shift right=2, "\Downarrow y"] & r_0(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\ell_0(x) \arrow[dr, bend left, "\ell_1(x)"] \arrow[dr, bend right, "r_1(x)"'] \arrow[dr, phantom, "\Leftarrow x"] & \\
& r_0(x) \\
\ell_0(y) \arrow[ur, bend left, "\ell_1(y)"] \arrow[ur, bend right, "r_1(y)"'] \arrow[ur, phantom, "\Rightarrow y"] &
\end{tikzcd}$$

The fact that these homomorphism axioms, as well as the interchange axiom, are given by inclusions rather than by equalities thus encodes the behaviour of composition and source/target maps with respect to the various shapes we can build from higher globular cells.

10.3.1. Remark. The n-catoid axioms contain redundancy. The proof assistant Isabelle's SAT-solvers and first-order equational theorem provers have been used for an analysis. For irredundancy of a formula φ with respect to a set A of formulas, we ask the SAT solvers for a model of $A \cup \{\neg\varphi\}$. For redundancy, we ask the theorem provers for a proof of $A \vdash \varphi$. This often succeeds in practice. Because of the set-up of n-catoids in terms of pairs of 2-catoids, an analysis of 2-catoids suffices.

10.3.2. Proposition. *The following n-catoid axioms are irredundant and imply the other axioms shown above:*

$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z),$$
$$\ell_j(x \odot_i y) = \ell_j(x) \odot_i \ell_j(y), \qquad r_j(x \odot_i y) = r_j(x) \odot_i r_j(y).$$

This reduction is convenient for relating structures, as in our correspondence results below, and for theorem proving. More generally, the single-set approach makes n-categories accessible to SMT solvers and first-order automated theorem provers.

10.4. GLOBULAR n-QUANTALES

In this section we define the quantalic structures that match the (globular) n-catoids in Jónsson-Tarski style duality and modal correspondence results. These are in particular globular n-Kleene algebras, as defined in Section 9.1.

A *(globular) n-quantale* is a structure $(Q, \leq, \cdot_i, 1_i, d_i, r_i)_{0 \leq i < n}$ such that for each $0 \leq i < n$, the structure $(Q, \leq, \cdot_i, 1_i, d_i, r_i)$ is a modal quantale. Furthermore, we require that the structures interact, for all $0 \leq i < j < n$, via

$$(w \cdot_j x) \cdot_i (y \cdot_j z) \leq (w \cdot_i y) \cdot_j (x \cdot_i z),$$
$$d_j(x \cdot_i y) = d_j(x) \cdot_i d_j(y), \qquad r_j(x \cdot_i y) = r_j(x) \cdot_i r_j(y),$$
$$d_i(x \cdot_j y) \leq d_i(x) \cdot_j d_i(y), \qquad r_i(x \cdot_j y) \leq r_i(x) \cdot_j r_i(y),$$
$$d_j(d_i(x)) = d_i(x).$$

The axiomatisation of n-quantales is already reduced and irredundant in the sense described in Remark 10.3.1, and is inspired by that of globular n-Kleene algebras, see Section 9.1. The axioms from globular HKA missing from the above list are consequences of the reduced axioms, see Lemma 10.4.1 below.

The axioms of n-quantales and n-catoids show a mismatch: $d_j \circ d_i = d_i$ is an n-quantale axiom while $\ell_j \circ \ell_i = \ell_i$ is derivable in n-catoids. The same holds for the two weak homomorphism axioms of n-quantales. For $d_j \circ d_i = d_i$ this can be explained as follows. Our proof of $\ell_j \circ \ell_i = \ell_i$ relies on $\Delta(x, y) \Rightarrow r(x) = \ell(y)$, but Remark 10.2.1 explains that the corresponding property is not available for quantales. Further, the related property $xy \neq \bot \Rightarrow r(x) \wedge d(y) \neq \bot$ is too weak to translate the proof of $\ell_j \circ \ell_i = \ell_i$ to quantales.

10.4.1. Lemma. *In every n-quantale, for $0 \leq i < j < n$,*

i) $d_j \circ r_i = r_i$, $r_j \circ d_i = d_i$ *and* $r_j \circ r_i = r_i$,

ii) $1_j \cdot_i 1_j = 1_j$ *and* $1_i \cdot_j 1_i = 1_i$,

iii) $1_i \leq 1_j$,

iv) $d_j(1_i) = 1_i$, $d_j(1_i) = 1_i$, $r_j(1_i) = 1_i$ *and* $r_j(1_i) = 1_i$,

v) $d_i \circ d_j = d_j \circ d_i$, $d_i \circ r_j = r_j \circ d_i$, $r_i \circ d_j = d_j \circ r_i$ *and* $r_i \circ r_j = r_j \circ r_i$,

vi) $d_i(x \cdot_j y) = d_i(x \cdot_j d_j(y))$ *and* $r_i(x \cdot_j y) = r_i(r_j(x) \cdot_j y)$.

10.4.2. Remark. In two dimensions, the interchange laws

$$(w \cdot_j x) \cdot_i (y \cdot_j z) \leq (w \cdot_i y) \cdot_j (x \cdot_i z)$$

of n-quantales appear in concurrent semirings [48] and concurrent Kleene algebras and quantales [75]. They have often been contrasted with the seemingly equational interchange laws of categories. Yet this ignores the subtle nature of equality in categories, which may depend on definedness of terms in equations. The homomorphism axioms of globular n-catoids encode such definedness presuppositions explicitly. As we have seen, this sometimes leads to inclusions and sometimes to equations.

Finally we consider the interactions of the Kleene stars with the n-structure.

10.4.3. Lemma. *In every n-quantale, for $0 \leq i < j < n$,*

i) $d_k(x) \cdot_i y^{*_j} \leq (d_k(x) \cdot_i y)^{*_j}$ *and* $x^{*_j} \cdot_i r_k(y) \leq (x \cdot_i r_k(y))^{*_j}$ *for* $k \in \{i, j\}$,

ii) $(x \cdot_j y)^{*_i} \leq x^{*_i} \cdot_j y^{*_i}$.

The properties in **i)** feature as axioms of globular n-Kleene algebras in [17], as recalled in Section 9.1.16, while **ii)** is already present in HKA, see Proposition 9.1.18. In sum, together with laws in previous sections all axioms of globular n-Kleene algebras have now been derived from the smaller set of axioms for n-quantales.

10.5. Correspondences for Power-set Quantales

Now that we have introduced the algebraic structures we need, we can state the correspondence theorem.

10.5.1. Theorem (Correspondence theorem for power-set n-quantales).

i) *Let X be a local n-catoid. Then $(\mathcal{P}X, \subseteq, \odot_i, E_i, \ell_i, r_i)_{0 \leq i < n}$ is an n-quantale.*

ii) *Let $\mathcal{P}X$ be an n-quantale in which $E_i \neq \emptyset$. Then X is a local n-catoid.*

Specialising to the case of n-categories, we obtain the following corollary:

10.5.2. Corollary. *Every n-category lifts to an n-quantale.*

Recall that Proposition 9.1.20, first found in [17], showed one direction of this correspondence in the case of free higher categories. Indeed, it states that lifting the free category generated by an n-polygraph equipped with a cellular extension results in a globular n-Kleene algebra. Since, in the case of the proposition, the latter is a power-set algebra, it is in particular an n-quantale. The correspondence provides a solid mathematical foundation for the axioms given for HKA in [17]. Furthermore, the n-quantalic approach englobes most cases of interest and provides a more streamlined axiomatisation.

A classical result by Gautam [47] shows that equations lift to the power-set level if each variable in the equation occurs precisely once in each side. These results have later been generalised by Grätzer and Whitney [63], or see [9] for an overview.

It is therefore no surprise that all the (unreduced) axioms of n-catoids lift directly to corresponding properties that we have already derived from the globular n-quantale axioms in Lemma 10.4.1. Nevertheless Gautam's result does not prima facie cover multioperations or even constructions of convolution algebras.

Jónsson and Tarski have considered relation algebra based on Boolean algebras instead of complete lattice. This makes no difference. See [74, 90] for details.

PART II.
TOPOLOGY OF CALCULATION

CHAPTER 11.
PRELIMINARIES

This chapter introduces notions from category theory and directed topology, as well as persistence theory, which are necessary for the treatment of the time-reversal problem for natural homotopy, addressed in Chapter 12, and the relationship between natural homology and persistent homology, see Chapter 13.

First, in Section 11.1, we recall the notion of group object and the fixed object slice category, before describing the structural properties of group objects therein [93], see Section 11.1.3. We then recall the definitions of natural systems, first appearing in the cohomology theory of small categories [5], and an augmentation thereof called lax systems in Section 11.2. The latter are due to Porter [93], and combine the notion of natural system with the structure of a lax functor. Section 11.3 describes how this extra structure defines a composition pairing on a natural system, and that lax systems are equivalent to these objects. Tying all of this together, in Section 11.4 we show that natural systems with composition pairings are equivalent to group objects in the corresponding fixed object slice category.

In Section 11.5, we recall from [62] the definitions of semi-exact and homological categories, exact sequences and show that the categories of groups and of pointed sets can be embedded in the category of actions, thus providing a common codomain for natural homotopy functors of all dimensions. This results in Proposition 11.5.7, a consequence of a result from [62], in which we show that we obtain a long exact sequence in the category of actions. In Section 11.6, we recall notions of natural homotopy and natural homology. For this, we first recall the notion of directed space, see [43, 61], and then define the invariants, first introduced in [31]. Finally, in Section 11.7, we recall the basics of persistence theory [19].

This chapter contains no original material and is included for completeness purposes.

11.1. GROUP OBJECTS

In this section we recall standard definitions for group objects and slice categories, see for example [89] for more information. In Section 11.1.3 we recall results regarding the structure of group objects in a certain slice category. These notions are from working notes of Porter [93] concerning the homology of small categories.

11.1.1. Fixed-object slice category.

We denote by $\mathbf{Cat}$ the category of small categories. Given some set Σ_0, consider $\mathbf{Cat}_{\Sigma_0}$, the subcategory of $\mathbf{Cat}$ consisting of those small categories with Σ_0 as their object set, and in which we only take functors which are the identity on 0-cells. Given some category $\mathcal{B} \in \mathbf{Cat}_{\mathcal{B}_0}$, we denote by $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ the comma category of objects in $\mathbf{Cat}_{\mathcal{B}_0}$ over $\mathcal{B}$.

Thus, the objects are pairs $(\mathcal{C}, p)$, where $\mathcal{C} \in \mathbf{Cat}_{\mathcal{B}_0}$ and $p : \mathcal{C} \to \mathcal{B}$ is a functor preserving the common object set Σ_0. Arrows $f : (\mathcal{C}, p) \to (\mathcal{C}', p')$ are those of $\mathbf{Cat}_{\mathcal{B}_0}$, preserving 0-cells, such that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{C}' \arrow[dl, "p'"] \\
& \mathcal{B} &
\end{tikzcd}$$

is commutative in $\mathbf{Cat}_{\mathcal{B}_0}$.

Note that $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ has arbitrary limits, and that its terminal object is the pair $(\mathcal{B}, id_{\mathcal{B}})$. Given an object $(\mathcal{C}, p)$ in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ and an arrow $f : x \to y$ of $\mathcal{B}$, the pre-image of f in $\mathcal{C}$ under p is the set

$$\mathcal{C}_f := \{c : x \to y \in \mathcal{C} \ \mid \ p(c) = f\},$$

called the *fibre of f under p*.

11.1.2. Group objects.

Here we recall the definition of group objects in a category. Consider an arbitrary category $\mathcal{A}$ having finite products. We will denote by $\top$ the empty product, which is thus the terminal object in $\mathcal{A}$. In such a category, a *group object*, also called *internal group*, is a tuple $(G, \mu, \eta, (\cdot)^{-1})$, where G is an object of $\mathcal{A}$, and

$$\mu : G \times G \to G \qquad \eta : T \to G \qquad (\cdot)^{-1} : G \to G$$

are morphisms of $\mathcal{A}$, which are interpreted, respectively, as group multiplication, the identity element, and the inversion map, in the sense that these morphisms must satisfy the group axioms, expressed in terms of the commutativity of certain diagrams in $\mathcal{A}$. For example, the following express that η picks out the (left and right) identity, and that μ is an associative operation:

$$\begin{tikzcd}
\top \times G \arrow[r, "\eta \times 1_G"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \arrow[d, "\mu"'] & G \times \top \arrow[l, "1_G \times \eta"'] \arrow[dl, "\cong"] \\
G &
\end{tikzcd}$$

$$\begin{tikzcd}
(G \times G) \times G \arrow[r, "\cong"] \arrow[dr, "\mu \times 1_G"'] & G \times (G \times G) \arrow[r, "1_G \times \mu"] & G \times G \arrow[d, "\mu"] \\
 & G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

Morally, a group object G in a category $\mathcal{A}$ is thus a group structure on G encoded by morphisms of $\mathcal{A}$. For a more precise definition, we refer the reader to [89, III.6].

We can also define a notion of *group object homomorphism* between group objects G and G'. These are morphisms f of $\mathcal{A}$ such that the diagrams encoding homomorphism properties, namely,

$$\begin{tikzcd}
G \times G \arrow[r, "\mu"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] & \top \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & G \arrow[d, "f"] \\
G' \times G' \arrow[r, "\mu'"] & G' & & G'
\end{tikzcd}$$

commute in $\mathcal{A}$. Taking these as morphisms, we can define the category $\mathbf{Gp}(\mathcal{A})$ of group objects in $\mathcal{A}$.

Adding a diagram expressing commutativity of μ, namely:

$$\begin{tikzcd}
 & G & \\
G \times G \arrow[ur, "\mu"] \arrow[rr, "\tau"'] & & G \times G \arrow[ul, "\mu"']
\end{tikzcd}$$

where τ exchanges the factors of the product, we can define *abelian group objects* in $\mathcal{A}$, the category of which will be denoted $\mathbf{Ab}(\mathcal{A})$.

11.1.3. Group objects in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. We now turn to the case of group objects in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. We will see that for a group object $(\mathcal{C}, p)$ of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, the group structure descends to the fibres $\mathcal{C}_f$ above each arrow f of B. First note that if $((\mathcal{C}, p), \mu, \eta, (-)^{-1})$ is a group object in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, since $(\mathcal{B}, id_{\mathcal{B}})$ is the terminal object, we have a morphism $\eta : (\mathcal{B}, id_{\mathcal{B}}) \to (\mathcal{C}, p)$, *i.e.* the following diagram is commutative in $\mathbf{Cat}_{\mathcal{B}_0}$:

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "\eta"] \arrow[dr, "id_{\mathcal{B}}"'] & & \mathcal{C} \arrow[dl, "p"] \\
 & \mathcal{B} &
\end{tikzcd}$$

This implies not only that every fibre $\mathcal{C}_f$ is non-empty, since $\eta(f) \in \mathcal{C}_f$, but additionally that η splits p in $\mathbf{Cat}_{\mathcal{B}_0}$. Therefore each hom-set $\mathcal{C}(x, y)$ is the coproduct (*i.e.* disjoint union) of the fibres:

$$\mathcal{C}(x,y) = \coprod_{f \in \mathcal{B}(x,y)} \mathcal{C}_f$$

We will denote elements of this set (c, f) where $c \in C_f$, but will sometimes simply write c when no confusion is possible.

The product in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ of an object $(\mathcal{C}, p)$ with itself is given by a pullback over $\mathcal{B}$, and is denoted $(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \tilde{p})$. The category $\mathcal{C} \times_{\mathcal{B}} \mathcal{C}$ has 0-cells Σ_0 and for $x, y \in \Sigma_0$,

$$(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})(x,y) = \{(c,d) \in \mathcal{C}(x,y)^2 \mid p(c) = p(d)\},$$

and $\tilde{p}$ assigns to each pair (c, d) of 1-cells in $\mathcal{C} \times_{\mathcal{B}} \mathcal{C}$ their common image under p. These hom-sets thus split as above, giving a decomposition in (set) products of fibres:

$$(\mathcal{C} \times_{\mathcal{B}} \mathcal{C})(x,y) = \coprod_{f \in \mathcal{B}(x,y)} \mathcal{C}_f \times \mathcal{C}_f.$$

By definition of μ, we have that the following diagram commutes in $\mathbf{Cat}_{\mathcal{B}_0}$:

$$\begin{tikzcd}
\mathcal{C} \times_{\mathcal{B}} \mathcal{C} \arrow[rr, "\mu"] \arrow[dr, "\tilde{p}"'] & & \mathcal{C} \arrow[dl, "p"] \\
& \mathcal{B} &
\end{tikzcd}$$

If $c, d \in \mathcal{C}_f$, then since $f = \tilde{p}(c,d) = p(\mu(c,d))$, we have that $\mu(c,d) \in \mathcal{C}_f$. Thus, the fibres are preserved by μ, and there is an induced map μ_f for each arrow f of $\mathcal{B}$,

$$\mu_f : \mathcal{C}_f \times \mathcal{C}_f \to \mathcal{C}_f.$$

Furthermore, this endows each fibre with a group structure. Indeed, examining the commutative diagrams satisfied by group objects, it is routinely verified that $(\mathcal{C}_f, \mu_f, \eta(f))$ is a group.

11.1.4. Remark. This structure allows us to interpret $\mathcal{C}$ as a category enriched in groupoids, *i.e.* a (2,1)-category. Indeed, viewing elements $c \in \mathcal{C}_f$ as 2-cells and defining vertical composition by $c \star_1 d := \mu_f(c,d)$, we coherently define a 2-category by functoriality of μ. In this interpretation, for objects $x, y \in \Sigma_0$, the hom-category $\mathcal{C}(x,y)$ is a discrete groupoid, its 0-cells being 1-cells $f : x \to y$ of $\mathcal{B}$, and in which the hom-set $\mathcal{C}(x,y)(f,g)$ empty for $f \neq g$ and equal to $\mathcal{C}_f$ otherwise. This interpretation of group objects in the fixeed objecet slice category is due to Porter [93].

11.1.5. The opposite group. The *opposite group* of an internal group $((\mathcal{C}, p), \mu, \eta, (-)^{-1})$ in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ is the internal group $(\mathcal{C}^o, p^o)$ in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}^o$, for which the multiplication, identity and inverse maps, denoted respectively by μ^o, η^o and $(-)^{-1}_o$, are the induced opposite maps of μ, η and $(-)^{-1}$. Note that the fibre group $\mathcal{C}_f$ in $\mathcal{C}$ associated to a 1-cell f of $\mathcal{B}$ is equal to the fibre group $\mathcal{C}^o_{f^o}$ associated to its opposite f^o.

11.2. NATURAL AND LAX SYSTEMS

Now we define natural systems and lax systems on a category, and see that the two notions are related; a natural system equipped with some extra structure is equivalent to a lax system. Results and definitions in this section are due to Porter [93].

11.2.1. Natural systems. Given a category $\mathcal{B}$, we consider the category of factorisation of $\mathcal{B}$, denoted $F\mathcal{B}$, in which 0-cells are 1-cells of $\mathcal{B}$, and in which 1-cells from f to f' are *extensions* (u,v), *i.e.* pairs of 1-cells of $\mathcal{B}$ such that $ufv = f'$. Composition is given by

$$(u,v)(u',v') = (u'u, vv'),$$

whenever u and v are composable with u' and v', respectively. The identity at $f : x \to y$ is the pair $(1_x, 1_y)$.

We also define subcategories $\mathcal{RB}$ and $\mathcal{LB}$ of $F\mathcal{B}$, having the same 0-cells as $F\mathcal{B}$, but taking only extensions of the form $(1,v)$ or $(u,1)$, respectively. $\mathcal{RB}$ and $\mathcal{LB}$ generate the factorisation category; for more information on these subcategories, we refer the reader to [116].

A *natural system on a category* $\mathcal{C}$ *with values in a category* $\mathbf{V}$ is a functor

$$D : F\mathcal{C} \to \mathbf{V}.$$

We will denote by D_f (resp. $D(u,v)$) the image of a 0-cell f (resp. 1-cell (u,v)) of $F\mathcal{C}$. In most cases, we will consider natural systems with values in the category $\mathbf{Set}_*$ of pointed sets, the category $\mathbf{Gp}$ of groups, the subcategory $\mathbf{Ab}$ of abelian groups, or the category $\mathbf{Act}$, then called *natural systems of pointed sets*, *of groups*, *of abelian groups*, or *of actions* respectively.

We denote by $\mathbf{NatSys}(\mathcal{C},\mathbf{V})$ the category whose objects are natural systems on $\mathcal{C}$ with values in $\mathbf{V}$ and in which morphisms are natural transformations between functors. The category of natural systems with values in $\mathbf{V}$, denoted by $\mathbf{opNat}(\mathbf{V})$, is defined as follows:

i) its objects are the pairs $(\mathcal{C}, D)$ where $\mathcal{C}$ is a category and D is a natural system on $\mathcal{C}$ with values in $\mathbf{V}$,

ii) its morphisms are pairs

$$(\Phi,\tau) : (\mathcal{C}, D) \to (\mathcal{C}', D')$$

consisting of a functor $\Phi : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\tau : D \to \Phi^* D'$, where the natural system $\Phi^* D' : F\mathcal{C} \to \mathbf{V}$ is defined by

$$(\Phi^* D')(f) = D'(\Phi f),$$

for any 1-cell f in $\mathcal{C}$ and $\Phi^* D'(u,v) = D'(\Phi(u), \Phi(v))$ for any 1-cells u and v in $\mathcal{C}$,

iii) composition of morphisms (Ψ,σ) with (Φ,τ) is defined by

$$(\Psi,\sigma)\circ(\Phi,\tau) = (\Psi\circ\Phi, (\Phi^*\sigma)\circ\tau).$$

Lax systems extend the notion of natural systems. In order to define them, we must first define lax functors and suspensions of categories.

11.2.2. Lax functors. We recall that given two 2-categories $\mathcal{M}$ and $\mathcal{N}$, a *lax functor* from $\mathcal{M}$ to $\mathcal{N}$ is a data consisting of

- **i)** A map $F : \mathcal{M}_0 \to \mathcal{N}_0$,
- **ii)** A functor $F_{x,y} : \mathcal{M}(x, y) \to \mathcal{N}(F(x), F(y))$ of hom-categories for all 0-cells x, y in $\mathcal{M}$,
- **iii)** A 2-cell $c_{f,g} : F_{x,y}(f)F_{y,z}(g) \Rightarrow F_{x,z}(fg)$ of $\mathcal{N}$, for each pair of composable 1-cells f and g of $\mathcal{M}$, where the juxtaposition on the left (resp. right) side denotes the composition $\star_0^{\mathcal{N}}$ (resp. $\star_0^{\mathcal{M}}$),
- **iv)** A 2-cell $c_x : 1_{F(x)} \Rightarrow F(1_x)$ of $\mathcal{N}$ for each 0-cell x of $\mathcal{M}$.

These assignments must satisfy the following three conditions:

1) *The naturality condition:* the assignment $(f, g) \mapsto c_{f,g}$ is natural in (f, g), in the sense that c is a natural transformation between functors induced by the $F_{x,y}$, namely those corresponding to the clockwise and anticlockwise composites in the following diagram:

$$\begin{tikzcd}
\mathcal{M}(x,y) \times \mathcal{M}(y,z) \arrow[r, "F_{x,y} \times F_{y,z}"] \arrow[d, "\star_0^{\mathcal{M}}"'] & \mathcal{N}(F(x),F(y)) \times \mathcal{N}(F(y),F(z)) \arrow[d, "\star_0^{\mathcal{N}}"] \\
\mathcal{M}(x,z) \arrow[r, "F_{x,z}"'] & \mathcal{N}(F(x),F(z))
\end{tikzcd}$$

2) *The cocycle condition:* for 1-cells f, g and h such that the composite fgh is defined, the following diagram commutes in $\mathcal{N}$

$$\begin{tikzcd}
F(f)F(g)F(h) \arrow[r, Rightarrow, "c_{f,g}F(h)"] \arrow[d, Rightarrow, "F(f)c_{g,h}"'] & F(fg)F(h) \arrow[d, Rightarrow, "c_{fg,h}"] \\
F(f)F(gh) \arrow[r, Rightarrow, "c_{f,gh}"'] & F(fgh)
\end{tikzcd}$$

3) *The left and right unit conditions:* for every 1-cell $f : x \to y$ of $\mathcal{M}$ the following diagrams commute in $\mathcal{N}$:

$$\begin{tikzcd}
F(1_x)F(f) \arrow[r, Rightarrow, "c_{1_x,f}"] & F(f) = F(1_xf) \\
1_{F(x)}F(f) = F(f) \arrow[u, Rightarrow, "c_xF(f)"] \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(f1_y) = F(f) & F(f)F(1_y) \arrow[l, Rightarrow, "c_{f,1_y}"'] \\
& F(f) = F(f)1_{F(y)} \arrow[u, Rightarrow, "F(f)c_y"'] \arrow[ul, equal]
\end{tikzcd}$$

11.2.3. Remark. The naturality of c in (f,g) requires that if $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ are 2-cells of $\mathcal{M}$, there is a commutative diagram in $\mathcal{N}$:

$$\begin{tikzcd}
F_{x,y}(f)F_{y,z}(g) \arrow[r, Rightarrow, "c_{f,g}"] \arrow[d, Rightarrow, "F_{x,y}(\alpha)F_{y,z}(\beta)"'] & F_{x,z}(fg) \arrow[d, Rightarrow, "F_{x,z}(\alpha\beta)"] \\
F_{x',y'}(f')F_{y',z'}(g) \arrow[r, Rightarrow, "c_{f',g'}"'] & F_{x',z'}(f'g')
\end{tikzcd}$$

The transformation c thus makes F homotopy-equivalent to a 2-functor in the sense that it provides F with a weakened functorial behaviour with respect to 0-composition of 2-cells. Note that if $\mathcal{M}$ has only identity 2-cells, the naturality condition only requires the existence of assignments $(f,g) \mapsto (c_{f,g} : F_{x,y}(f)F_{y,z}(g) \Rightarrow F_{x,z}(fg))$ for all composable pairs (f,g) of 1-cells, and $x \mapsto (c_x : 1_{F(x)} \Rightarrow F(1_x))$ for 0-cells.

11.2.4. Suspension of monoidal categories. A *monoidal category* is a triple $(\mathcal{C}, \otimes, I)$, with $\mathcal{C}$ a category, $\otimes$ a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

and I an object of $\mathcal{C}$. This structure comes equipped with natural isomorphisms

$$\alpha : ((\cdot) \otimes (\cdot)) \otimes (\cdot) \xrightarrow{\cong} (\cdot) \otimes ((\cdot) \otimes (\cdot))$$

$$\lambda : I \otimes (\cdot) \xrightarrow{\cong} id_{\mathcal{C}} \qquad \rho : (\cdot) \otimes I \xrightarrow{\cong} id_{\mathcal{C}}$$

expressing associativity of $\otimes$, and that I is its left and right identity. When these natural isomorphisms are all identities, we say that $\mathcal{C}$ is a *strict* monoidal category. We say that $\mathcal{C}$ is *symmetric* when $\otimes$ is a commutative operation, in the sense that there exists a natural isomorphism

$$\beta : \otimes \longrightarrow \otimes \circ \tau,$$

where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the functor which exchanges the order of factors, such that for all objects A and B,

$$\beta_{A,B} \circ \beta_{B,A} = 1_{A \otimes B}.$$

These natural isomorphisms must satisfy some coherence relations which we will not list here; we invite the interested reader to consult [89, VII.1].

We will place ourselves in the setting of strict monoidal categories to assure that $\otimes$ satisfies the axioms for composition of 1-cells in a category. This, as well as an interchange law between $\otimes$ and composition, automatic by functoriality of $\otimes$, are what we will need to coherently define suspensions.

Given a strict monoidal category $(\mathcal{C}, \otimes, I)$, we define its *suspension* $\mathcal{C}[1]$ as the 2-category with:

- A single 0-cell, $*$.
- As 1-cells, the 0-cells of $\mathcal{C}$, with 0-composition given by $\otimes$.
- As 2-cells, the 1-cells of $\mathcal{C}$, with 1-composition given by the composition in $\mathcal{C}$.

This construction is an example of horizontal categorification. A well known example of this is the realisation of a monoid as a category with a single object.

11.2.5. Strictification. Note that if $\mathcal{C}$ is not a strict monoidal category, this construction does not define a 2-category. In this case the 0-composition is not associative, and the 1-cell 1_* is not really an identity; the structure we would obtain is called a bicategory. We will ignore this detail in what follows, treating all of the monoidal categories we encounter as strict, since a monoidal category can be made into a strict monoidal category in a way which is coherent with the 2-categorification of a bicategory. This strictification is not done by quotienting, but by a construction called *strictification*, which exploits the notion of clique in a category. The 2-categorification of a bicategory is in fact an adaption of the notion of strictification of a monoidal category. We refer the curious reader to [84, 88].

11.2.6. The suspension of Gp. In particular, we consider the suspension of the monoidal category $\mathbf{Gp}$, where $\otimes$ is given by the cartesian product of groups and the identity element is the trivial group I.

Let us describe this suspension in detail: $\mathbf{Gp}[1]$ has a single 0-cell $*$, 1-cells are groups G, H, their composition being $G \times H =: G \star_0 H$, and 2-cells are group homomorphisms ϕ, ψ, with 1-composition given by the usual composition of homomorphisms, but with the order inversed, *i.e.* $\phi \star_1 \psi = \psi \circ \phi$. The 0-composition is given by $\phi \star_0 \psi = \phi \times \psi$. This defines a bicategory, but we will ignore this, and think of $\mathbf{Gp}[1]$ as a 2-category, obtained as the suspension of the strictification of $\mathbf{Gp}$. We can similarly obtain 2-categories from the suspensions of other (non-strict) monoidal categories of interest, such as $\mathbf{Ab}$ or $\mathbf{Set}_*$.

11.2.7. Lax systems. We recall from [93] the notion of lax system on a category $\mathcal{B}$ with values in a cartesian category $(\mathbf{V}, \times, T)$, where T is the terminal object of $\mathbf{V}$.

A 2-category is said to be *locally discrete* when its hom-categories are discrete, *i.e.* have only identity arrows. We can interpret a category $\mathcal{B}$ as a locally discrete 2-category $\mathcal{B}^2$ by adding an identity 2-cell for each 1-cell of $\mathcal{B}$. Moreover, the cartesian category $\mathbf{V}$ may be suspended into a 2-category, which we denote by $\mathbf{V}[1]$. The 2-category $\mathbf{V}[1]$ has only one 0-cell, and its 1-cells and their 0-composites correspond to 0-cells in $\mathbf{V}$ and their products, while its 2-cells correspond to the 1-cells of $\mathbf{V}$. We recall from [93] that a *lax system* on a category $\mathcal{B}$ with values in a cartesian category $\mathbf{V}$ is a lax functor from $\mathcal{B}^2$ with values in $\mathbf{V}[1]$.

It is shown in [93] that given a lax system (F, c) on $\mathcal{B}$ with values in $\mathbf{V}$, we can construct a natural system UD by associating to each 0-cell f of $F\mathcal{B}$ the 0-cell D_f of $\mathbf{V}$, and to each 1-cell (u, v) of $F\mathcal{B}$ the 1-cell $D_f \to D_{ufv}$ sending d to $c_{uf,v}(c_{u,f}(1, d), 1)$. We define the *category of lax systems on $\mathcal{B}$ with values in* $\mathbf{V}$, denoted by $\mathbf{LaxSys}(\mathcal{B}, \mathbf{V})$, in which a morphism from (F, c) to (F', c') is a natural transformation $\alpha : UD \Rightarrow UD'$ between the corresponding underlying natural systems, such that the following diagram commutes:

$$\begin{tikzcd}
UD_f \times UD_g \arrow[r, "c_{f,g}"] \arrow[d, "\alpha_f \times \alpha_g"'] & UD_f \arrow[d, "\alpha_{fg}"] \\
UD'_f \times UD'_g \arrow[r, "c'_{f,g}"'] & UD'_{fg}
\end{tikzcd}$$

Notice that, as opposed to the case of natural systems, the domain of the lax functor is not the factorisation category of $\mathcal{B}$. This is because the dimension shift, in the case of lax systems, instead takes place in the codomain, by means of suspension. We will use the notation D_f instead of $D(f)$, and relax the notation for lax functors in the case of lax systems, omitting the specification $D_{x,y}$ on D since $\mathbf{V}[1]$ only has one object.

11.2.8. Lax systems of groups. Let us make explicit what a lax system of groups on a category $\mathcal{B}$ looks like. We will write $\phi : G \to G'$ for group homomorphisms, *i.e.* 2-cells of $\mathbf{Gp}[1]$. Firstly, the object set of $\mathcal{B}$ is collapsed, all 0-cells being sent to $*$. Each 1-cell $f : x \to y$ of $\mathcal{B}$ is associated to a group D_f, and composable arrows f and g give a 2-cell of $\mathbf{Gp}[1]$, so a group homomorphism, $\nu_{f,g} : D_f \times D_g \to D_{fg}$. Recall from Remark 11.2.3 that since $\mathcal{B}^2$ is locally discrete, the naturality condition requires nothing more of this assignment. Furthermore, since I is the trivial group, we have no choice in defining the homomorphisms $\nu_x : 1_* = I \to D_{1_x}$ associated to objects.

As a consequence of the unit diagrams,

$$\begin{tikzcd}
D_f & D_f \times D_{1_y} \arrow[l, "\nu_{f,1_y}"'] & D_{1_x} \times D_f \arrow[r, "\nu_{1_x,f}"] & D_f \\
& D_f \times I \arrow[ul, "\cong"] \arrow[u, "1_{D_f} \times \nu_y"'] & I \times D_f, \arrow[u, "\nu_x \times 1_{D_f}"] \arrow[ur, "\cong"'] & .
\end{tikzcd}$$

we have the following identities:

$$\nu_{f,1_y}(d, 1) = d = \nu_{1_x,f}(1, d).$$

The cocycle condition, for a triple (f, g, h) such that the composite fgh is defined, is the commutativity of the following diagram:

$$\begin{tikzcd}
D_f \times D_g \times D_h \arrow[r, "\nu_{f,g} \times id_{D_h}"] \arrow[d, "id_{D_f} \times \nu_{g,h}"'] \arrow[dr, dotted] & D_{fg} \times D_h \arrow[d, "\nu_{fg,h}"] \\
D_f \times D_{gh} \arrow[r, "\nu_{f,gh}"] & D_{fgh}
\end{tikzcd}$$

This induces an unambiguous homomorphism $D_f \times D_g \times D_h \longrightarrow D_{fgh}$. With these observations, we establish the following relationship between lax and natural systems:

11.2.9. Proposition ([93]). *To a lax system of groups* (D, ν) *on a category* $\mathcal{B}$ *we can associate a natural system* $UD : F\mathcal{B} \to \mathbf{Gp}$. *We call this the underlying natural system of* (D, ν).

Proof. Clearly, to each 0-cell $f : x \to y$ of $F\mathcal{B}$ we associate the group D_f. For each extension $(u, v) : f \to f' = ufv$, we need to define a group homomorphism $D_f \longrightarrow D_{ufv}$. This can be done by restricting the homomorphism $D_u \times D_f \times D_v \longrightarrow D_{ufv}$, defined above by the cocycle condition, to the subgroup $\{1_{D_u}\} \times D_f \times \{1_{D_v}\}$. We will denote this homomorphism $UD(u, v)$. Explicitly, for $d \in D_f$, we have

$$UD(u, v)(d) = \nu_{uf,v}(\nu_{u,f}(1, d), 1) = \nu_{u,fv}(1, \nu_{f,v}(d, 1)).$$

Now all we need to do is verify that UD is a functor, *i.e.* that the assignment is compatible with composition. We first check that $UD(u', 1)UD(u, 1) = UD(u'u, 1)$; recall that due to the unit condition, $UD(u, 1)(d) = \nu_{u,f}(1, d)$ for all $d \in D_f$. Calculating the diagonal of the following cocycle diagram at $(1, 1, d)$,

$$\begin{tikzcd}
D_{u'} \times D_u \times D_f \arrow[r, "\nu_{u',u} \times 1_{D_f}"] \arrow[d, "1_{D'_u} \times \nu_{u,f}"'] & D_{u'u} \times D_f \arrow[d, "\nu_{u'u,f}"] \\
D_{u'} \times D_{uf} \arrow[r, "\nu_{u',uf}"'] & D_{u'uf}
\end{tikzcd}$$

we see that

$$\begin{aligned}
UD(u'u, 1)(d) &= \nu_{u'u,f}(1, d) \\
&= \nu_{u'u,f}(\nu_{u',u}(1, 1), d) = \nu_{u',uf}(1, \nu_{u,f}(1, d)) \\
&= UD(u', 1)(\nu_{u,f}(1, d)) = UD(u', 1)UD(u, 1)(d)
\end{aligned}$$

It can similarly be shown that $UD(1, v)UD(1, v') = UD(1, vv')$. Then, using that $UD(1, v)(d) = \nu_{f,v}(d, 1)$ and again that $D(u, 1)(d) = \nu_{u,f}(1, d)$, we see that the cocycle condition for (u, f, v) states exactly

$$D(1, v)D(u, 1) = D(u, v) = D(u, 1)D(1, v).$$

□

The above proposition can be reformulated in terms of a forgetful functor:

11.2.10. Proposition ([93]). *The assignment*

$$U : \mathbf{LaxSys}(\mathcal{B}, \mathbf{Gp}) \to \mathbf{NatSys}(\mathcal{B}, \mathbf{Gp})$$

is functorial.

Proof. This is immediate due to our choice of morphisms in $\mathbf{LaxSys}(\mathcal{B}, \mathbf{Gp})$, see Section 11.2.7. ☐

11.3. NATURAL SYSTEMS WITH COMPOSITION PAIRING

Here we describe how the extra structure provided by lax functors can be interpreted in natural systems. Results and definitions in this section are again from [93]. Given a natural system D on a category $\mathcal{B}$, we can begin to define a lax system by sending all 0-cells of $\mathcal{B}$ to $*$, and sending 1-cells f of $\mathcal{B}$ to the groups D_f given by the natural system. The only extra bit of structure provided by a lax system are the homomorphisms $c_{f,g}$.

11.3.1. Composition pairing. Let $\mathbf{V}$ be a category with finite products. Given a natural system D on a category $\mathcal{C}$ with values in $\mathbf{V}$, recall from [93] that a *composition pairing* associated to D consists of two families of morphisms of $\mathbf{V}$

$$\big(\, \nu_{f,g} : D_f \times D_g \to D_{fg} \,\big)_{f,g \in \mathcal{C}_1} \qquad \text{and} \qquad \big(\, \nu_x : T \to D_{1_x} \,\big)_{x \in \mathcal{C}_0},$$

where T is the terminal object in $\mathbf{V}$, such that the three following conditions are satisfied:

i) *naturality condition*: the following diagram

$$\begin{tikzcd} D_f \times D_g \arrow[r, "\nu_{f,g}"] \arrow[d, "D(u,1) \times D(1,v)"'] & D_{fg} \arrow[d, "D(u,v)"] \\ D_{uf} \times D_{gv} \arrow[r, "\nu_{uf,gv}"'] & D_{ufgv} \end{tikzcd}$$

commutes in $\mathbf{V}$ for all 1-cells f, g, u, v in $\mathcal{C}_1$ such that the composites are defined.

ii) *The cocycle condition:* the diagram

$$\begin{tikzcd} D_f \times D_g \times D_h \arrow[r, "\nu_{f,g} \times id_{D_h}"] \arrow[d, "id_{D_f} \times \nu_{g,h}"'] & D_{fg} \times D_h \arrow[d, "\nu_{fg,h}"] \\ D_f \times D_{gh} \arrow[r, "\nu_{f,gh}"'] & D_{fgh} \end{tikzcd}$$

commutes for all 1-cells f, g and h of $\mathcal{C}$ such that the composite fgh is defined,

iii) *The unit conditions:* the diagrams

$$\begin{tikzcd}
D_f & D_f \times D_{1_y} \arrow[l, "\nu_{f,1_y}"'] \\
 & D_f \times T \arrow[u, "1_{D_f} \times \nu_y"'] \arrow[ul, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
D_{1_x} \times D_f \arrow[r, "\nu_{1_x,f}"] & D_f \\
T \times D_f \arrow[u, "\nu_x \times 1_{D_f}"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

commute for every 1-cell $f : x \to y$ of $\mathcal{C}$.

11.3.2. Categeory of natural systems with composition pairings. The category of natural systems on $\mathcal{C}$ with values in $\mathbf{V}$ which admit a composition pairing is the category whose objects are pairs (D, ν), with D a natural system on $\mathcal{C}$ and ν a composition pairing associated to D. The morphims are natural transformations $\alpha : D \Rightarrow D'$ *compatible with the composition pairings* ν and ν', in the sense that the following diagram commutes in $\mathbf{V}$

$$\begin{tikzcd}
D_f \times D_g \arrow[r, "\nu_{f,g}"] \arrow[d, "\alpha_f \times \alpha_g"'] & D_{fg} \arrow[d, "\alpha_{fg}"] \\
D'_f \times D'_g \arrow[r, "\nu'_{f,g}"'] & D'_{fg}
\end{tikzcd}$$

for all composable 1-cells f and g in $\mathcal{C}$. We will denote this category of natural systems admitting a composition pairing by $\mathbf{NatSys}_\nu(\mathcal{C}, \mathbf{V})$. We denote by $\mathbf{opNat}_\nu(\mathbf{V})$ the subcategory of $\mathbf{opNat}(\mathbf{V})$ consisting of natural systems with values in $\mathbf{V}$ which admit a composition pairing, in which we take only those morphisms (Φ, τ), see Section 11.2.1, such that τ is compatible with the composition pairings ν and $\Phi^*\nu'$.

11.3.3. Remark. Here we interpret the naturality condition listed above for composition pairings. For this, we define the category $\mathbf{Pairs}(\mathcal{B})$ of pairs; its 0-cells are pairs $(f : x \to y, g : y \to z)$ of composable 1-cells of $\mathcal{B}$, and 1-cells are pairs $(u : x' \to x, v : z \to z')$ such that

$$(uf)(gv) = ufgv,$$

and therefore correspond to pairs of arrows $((u, 1), (1, v))$ in $\mathcal{L}\mathcal{B} \times \mathcal{R}\mathcal{B}$. This allows us to define functors

$$\begin{aligned}
P_1 : \mathbf{Pairs}(\mathcal{B}) &\longrightarrow \mathcal{L}\mathcal{B} \times \mathcal{R}\mathcal{B} \\
(f, g) &\longmapsto (f, g) \\
(u, v) &\longmapsto ((u, 1), (1, v))
\end{aligned}
\qquad\qquad
\begin{aligned}
P_2 : \mathbf{Pairs}(\mathcal{B}) &\longrightarrow F\mathcal{B} \\
(f, g) &\longmapsto fg \\
(u, v) &\longmapsto (u, v)
\end{aligned}$$

Composing these with $(\times) \circ (D, D)$ and D, respectively, where $\times$ is the cartesian product, *i.e.* the 0-composition in $\mathbf{V}[1]$, we obtain functors $D_\circ$ and ${}_\circ D$ respectively. Explicitly, we have

$$\begin{aligned} D_\circ : \mathbf{Pairs}(\mathcal{B}) &\longrightarrow \mathbf{V} \\ (f,g) &\longmapsto D_f \times D_g \\ (u,v) &\longmapsto D(u,1)\times D(1,v) \end{aligned} \qquad\qquad \begin{aligned} {}_\circ D : \mathbf{Pairs}(\mathcal{B}) &\longrightarrow \mathbf{V} \\ (f,g) &\longmapsto D_{fg} \\ (u,v) &\longmapsto D(u,v). \end{aligned}$$

Given a natural system D on a category $\mathcal{B}$, a composition pairing associated to D is a natural transformation $\nu : D_\circ \Rightarrow {}_\circ D$, which satisfies the cocycle and unit conditions.

This natural transformation represents a weak commutativity of 0-composition in $\mathcal{B}$ and 0-composition in $\mathbf{Gp}[1]$, as we would expect it to in the case of lax functors; it gives the character of a 2-functor up to homotopy equivalence. However, due to the "dimension juggling", this is reinterpreted in the context of the interplay between 0-composition in $\mathcal{B}$ and the right and left components $\mathcal{RB}$ and $\mathcal{LB}$ of $F\mathcal{B}$.

11.3.4. Commutator condition. Now to the question of which natural systems of groups admit a composition pairing. For all composable arrows f and g, the left and right actions of $F\mathcal{B}$ give us homomorphisms

$$\begin{tikzcd} D_f \arrow[r, "D(1,g)"] & D_{fg} & D_g \arrow[l, "D(f,1)"'] \end{tikzcd}$$

These define a homomorphism $\nu_{f,g} : D_f \times D_g \to D_{fg}$, letting

$$\nu_{f,g}(d,d') = D(f,1)(d').D(1,g)(d).$$

However, we could have decided to take $\tilde{\nu}_{f,g}(d,d') = D(1,g)(d).D(f,1)(d')$ instead. The requirement amounts to these two homomorphisms being equal, *i.e.* that D satisfies a commutation property for left and right morphisms of the factorisation category.

11.3.5. Proposition ([93]). *Let D be a natural system of groups on a category $\mathcal{B}$. Then*

$$D \textit{ admits a composition pairing } \nu \iff [D(f,1)(d'), D(1,g)(d)] = 1 \\ \forall d \in D_f, \forall d' \in D_g$$

and in both cases ν must be given by

$$\nu_{f,g}(d,d') = D(f,1)(d').D(1,g)(d) = D(1,g)(d).D(f,1)(d').$$

Proof. First suppose that D admits a pairing ν. Then for all (u,v), ν satisfies a naturality condition given by

$$\nu_{uf,gv} \circ (D(u,1)\times D(1,v)) = D(u,v)\circ \nu_{f,g}.$$

We examine this condition in the case of $(1,g)$:

$$\begin{tikzcd}
D_f \times D_1 \arrow[r, "\nu_{f,1}"] \arrow[d, "D(1,1) \times D(1,g)"'] & D_f \arrow[d, "D(1,g)"] \\
D_f \times D_g \arrow[r, "\nu_{f,g}"'] & D_{fg}
\end{tikzcd}$$

By this, and that $\nu_{f,1}(d,1) = d$, evaluation at $(d,1)$ gives

$$\nu_{f,g}(d,1) = \nu_{f,g} \circ (D(1,1) \times D(1,g))(d,1) = D(1,g) \circ \nu_{f,1}(d,1) = D(1,g)(d).$$

Similarly, we get $\nu_{f,g}(1,d') = D(f,1)(d')$. Then since $\nu_{f,g}$ is group homomorphism,

$$\nu_{f,g}(d,d') = D(f,1)(d').D(1,g)(d) = D(1,g)(d).D(f,1)(d'),$$

which proves the direct implication.

For the converse, we define $\nu_{f,g}(d,d') := D(f,1)(d').D(1,g)(d)$ and then check that this amounts to a composition pairing. This is almost immediate, since it is clearly a group homomorphism, and by calculation satisfies the naturality condition for pairings, as well as the unit conditions. The commutator hypothesis plays a role only in the verification of the cocycle condition. □

Note in particular that if a natural system is equipped with a composition pairing, the latter is unique. The commutation condition is useful for checking whether or not a given natural system admits a composition pairing; one only needs to check that $[D(f,1)(d), D(1,g)(d')] = 1$ for all $d \in D_g$ and $d' \in D_f$. We also get the following corollary:

11.3.6. Corollary ([93]). *Every natural system of abelian groups admits a composition pairing.*

11.3.7. Remark. The compatibility condition for natural transformations, see Section 11.3.2, is always satisfied in the case of natural systems of groups with composition pairings. Indeed, if α is a transformation of natural systems, we have

$$D'(1,g)(\alpha_f(d)) = \alpha_{fg}(D(1,g)(d)) \text{ and } D'(f,1)(\alpha_g(d')) = \alpha_{fg}(D(f,1)(d'))$$

for all $d \in D_f$ and $d' \in D_g$. Thus

$$\begin{aligned}
\alpha_{fg}(\nu_{f,g}(d,d')) =& \alpha_{fg}(D(1,g)(d).D(f,1)(d')) \\
=& \alpha_{fg}(D(1,g)(d)).\alpha_{fg}(D(f,1)(d')) \\
=& D'(1,g)(\alpha_f(d)).D'(f,1)(\alpha_g(d')) \\
=& \nu'_{f,g}(\alpha_f(d), \alpha_g(d')).
\end{aligned}$$

We thereby deduce that $\mathbf{NatSys}_\nu(\mathcal{B}, \mathbf{Gp})$ (resp. $\mathbf{opNat}_\nu(\mathbf{Gp})$) is a full subcategory of $\mathbf{NatSys}(\mathcal{B}, \mathbf{Gp})$ (resp. $\mathbf{opNat}(\mathbf{Gp})$), and that the categories $\mathbf{NatSys}(\mathcal{B}, \mathbf{Ab})$ and $\mathbf{NatSys}_\nu(\mathcal{B}, \mathbf{Ab})$ are equal.

11.3.8. Lax systems are natural systems with composition pairings. We now establish the equivalence of categories which links lax systems to natural systems with composition pairings.

11.3.9. Proposition ([93]). *There exist functors*

$$\Phi : \mathbf{NatSys}_\nu(\mathcal{B}, \mathbf{Gp}) \to \mathbf{LaxSys}(\mathcal{B}, \mathbf{Gp}) \qquad \Psi : \mathbf{LaxSys}(\mathcal{B}, \mathbf{Gp}) \to \mathbf{NatSys}_\nu(\mathcal{B}, \mathbf{Gp})$$

establishing an isomorphism of categories

$$\mathbf{LaxSys}(\mathcal{B}, \mathbf{Gp}) \cong \mathbf{NatSys}_\nu(\mathcal{B}, \mathbf{Gp}).$$

Proof. Given a natural system with composition pairing (D, ν), we take $\Phi(D, \nu)$ to be the lax system which sends 0-cells in $\mathcal{B}^2$ to $*$ and 1-cells f to D_f. The unit and cocyle conditions are satisfied by ν by default, and the naturality as required by the definition of lax functors is trivial since we view $\mathcal{B}^2$ is a locally discrete 2-category.

For a lax system $D = (D, \nu)$, we take $\Psi(D, \nu) = (UD, \nu)$. Recall from the proof of Proposition 11.2.9 we have $D(f, 1)(d') = \nu_{f,g}(1, d')$ and $D(1, g)(d) = \nu_{f,g}(d, 1)$. Thus

$$D(f, 1)(d').D(1, g)(d) = \nu_{f,g}(d, d') = D(1, g)(d).D(f, 1)(d'),$$

and we conclude that ν indeed defines a composition pairing via Proposition 11.3.5. All that is left is the cocycle and unit conditions, which are given by the lax structure.

The functoriality of these assignments is immediate from the choice of morphisms in each category. Furthermore, they are mutually inverse.

□

11.4. LAX SYSTEMS ARE GROUP OBJECTS

Here we describe the link between the group objects in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ and lax systems of groups on $\mathcal{B}$, or equivalently, natural systems on $\mathcal{B}$ with composition pairings. Indeed, in Theorem 11.4.3, we recall the isomorphism of categories

$$\mathbf{NatSys}_\nu(\mathcal{B}, \mathbf{Gp}) \simeq \mathbf{Gp}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}).$$

11.4.1. Group objects to natural systems with composition pairings. Starting with a group object $((\mathcal{C}, p), \mu, \eta, (-)^{-1})$, we construct a natural system of groups with composition pairing. Recall that each hom-set in $\mathcal{C}$ decomposes into fibres

$$\mathcal{C}(x, y) = \coprod_{f \in \mathcal{B}(x,y)} \mathcal{C}_f$$

and that each of these fibres carries a group structure $(\mathcal{C}_f, \mu_f, \eta(f))$. We define a natural system D with the assignment $f \mapsto \mathcal{C}_f$, *i.e.* $D_f = \mathcal{C}_f$. To a morphism $(u, v) : f \to ufv$

in $F\mathcal{B}$, we need to associate a group homomorphism $\mathcal{C}_f \to \mathcal{C}_{ufv}$. For $c \in \mathcal{C}_f$, this will be given by the assignment

$$c \longmapsto \eta(u) \star_0 c \star_0 \eta(v)$$

where $\star_0$ is the composition in $\mathcal{C}$. These are group homomorphisms by functoriality of μ and η. Furthermore, this assignment defines a functor $F\mathcal{B} \to \mathbf{Gp}$. Indeed, we have

$$\eta(u') \star_0 \eta(u) \star_0 c \star_0 \eta(v) \star_0 \eta(v') = \eta(u'u) \star_0 c \star_0 \eta(vv')$$

since η is a functor $\mathcal{B} \to \mathcal{C}$. We have defined a natural system D on $\mathcal{B}$.

Now we need to find the composition pairing associated to D. This will be given by the composition in $\mathcal{C}$; notice that by functoriality of μ, we have that for $c, c' \in \mathcal{C}_f$ and $d, d' \in \mathcal{C}_g$,

$$\mu_{fg}((c \star_0 d), (c' \star_0 d')) = (c \star_0 d) \star_1 (c' \star_0 d') = (c \star_1 c') \star_0 (d \star_1 d') = \mu_f(c, c') \star_0 \mu_g(d, d')$$

and thus $\star_0$, the composition in $\mathcal{C}$, provides a group homomorphism

$$\mathcal{C}_f \times \mathcal{C}_g \longrightarrow \mathcal{C}_{fg}.$$

It is this homomorphism we will use as the component of the composition pairing ν at (f, g). We still need to verify that ν is in fact a valid composition pairing. The calculation

$$\begin{aligned} c \star_0 d = \mu(c \star_0 d, \eta(fg)) &= \mu(c \star_0 d, \eta(f) \star_0 \eta(g)) \\ &= \mu(c \star_0 \eta(g), \eta(f) \star_0 d) \\ &= D(1, g)(c).D(f, 1)(d) \end{aligned}$$

gives us that $\nu_{f,g}(c, d) = D(1, g)(c).D(f, 1)(d)$, and by similar calculation we get

$$D(f, 1)(d).D(1, g)(c) = D(1, g)(c).D(f, 1)(d).$$

Proposition 11.3.5 then allows us to conclude that ν is indeed a composition pairing.

We have thus constructed a natural system with a pairing morphism from a group object of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. We will denote this assignment by $(D, \nu) =: \Gamma((\mathcal{C}, p), \mu, \eta, (-)^{-1})$.

11.4.2. Natural systems with composition pairings to group objects.

Now we will examine the inverse construction. From a natural system with composition pairing (D, ν), we start by defining the associated object $\mathcal{C}$ of $\mathbf{Cat}_{\mathcal{B}_0}$. As 0-cells, we of course take those of $\mathcal{B}$, and for 0-cells x and y, we define the associated hom-set by letting

$$\mathcal{C}(x, y) := \coprod_{f \in \mathcal{B}(x,y)} D_f.$$

Elements of these sets, *i.e.* 1-cells of $\mathcal{C}$, will be denoted by a pair (c, f) with $c \in D_f$. We need to define the composition $\star_0$ of such arrows, and this will be achieved using the composition pairing ν. A consequence of our definition of hom-sets is that we can decompose products as follows:

$$\mathcal{C}(x,y)\times\mathcal{C}(y,z) \quad = \coprod_{f\in B(x,y),g\in B(y,z)} D_f\times D_g.$$

This means we need a map

$$\coprod_{f\in B(x,y),g\in B(y,z)} D_f\times D_g \longrightarrow \coprod_{h\in B(x,z)} D_h.$$

We define composition fibre by fibre using $\nu_{f,g}: D_f\times D_g\to D_{fg}$, by setting

$$(c,f)\star_0(d,g) := (\nu_{f,g}(c,d), fg).$$

This defines a composition map for a category since it is associative by the cocycle condition, and has identities given, at x, by the pair $(1_{D_f}, 1_x^{\mathcal{B}})$.

Defining $p:\mathcal{C}\to\mathcal{B}$ as the identity on 0-cells, and assigning the pair (c,f) to f, gives, by definition of the composition in $\mathcal{C}$, a functor of $\mathbf{Cat}_{\mathcal{B}_0}$. Then $(\mathcal{C},p)$ is an object of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$; we still need to prove that it is a group object. For this, we must define μ and η.

We define $\eta:\mathcal{B}\to\mathcal{C}$ by setting $\eta(f):=(1_{D_f},f)$. By definition of composition in $\mathcal{C}$, this is a functor, and since $p\circ\eta = id_{\mathcal{B}}$, it is an arrow $(\mathcal{B}, id_{\mathcal{B}})\longrightarrow(\mathcal{C},p)$ of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. To define μ, recall that the product in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ is given by pullback over $\mathcal{B}$ in $\mathbf{Cat}_{\mathcal{B}_0}$:

$$(\mathcal{C},p)\times(\mathcal{C},p) = ((\mathcal{C}\times_{\mathcal{B}}\mathcal{C}),\tilde{p})$$

where

$$(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}) = \{((c,f),(d,g)) \mid p(c,f)=p(d,g)\ \textit{i.e. } f=g\} = \coprod_{f\in\mathcal{B}(x,y)} \mathcal{C}_f\times\mathcal{C}_f.$$

We can therefore let $\mu((c,f),(d,f)) := (c.d,f)$. This gives a functor of $\mathbf{Cat}_{\mathcal{B}_0}$ since

$$\begin{aligned}\mu((c,f)\star_0(d,g),(c',f)\star_0(d',g)) &= (\nu_{f,g}(c,d).\nu_{f,g}(c',d'), fg)\\ &= (\nu_{f,g}(c.c',d.d'), fg) = (c.c',f)\star_0(d.d',g)\\ &= \mu((c,f)(c',f))\star_0\mu((d,g),(d',g))\end{aligned}$$

where we used that $\nu_{f,g}$ is a group homomorphism. All that is left to check is the commutativity of the diagrams expressing the group axioms, but these are immediate since μ and η are defined in terms of the group structures on the fibres.

We will denote this assignment by $((\mathcal{C},p),\mu,\eta,(-)^{-1}) =: \Lambda(D,\nu)$.

11.4.3. Theorem ([93]). *The assignments Λ and Γ are functorial, and induce an isomorphism of categories*

$$\mathbf{NatSys}_{\nu}(\mathcal{B},\mathbf{Gp}) \simeq \mathbf{Gp}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}).$$

Proof. We first check that the above assignments are functorial.

Let $((\mathcal{C}, p), \mu, \eta, (-)^{-1})$ and $((\mathcal{C}', p'), \mu', \eta', [.]^{-1})$ be group objects in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, and ϕ a morphism between them; recall that ϕ is then a functor $(\mathcal{C}, p) \to (\mathcal{C}', p')$ satisfying the commutativity of certain diagrams expressing that ϕ behaves like a group homomorphism. Since for every $c \in \mathcal{C}_f$, $f = p(c) = p'(\phi(c))$, we have $\phi(f) \in \mathcal{C}'_f$. Thus ϕ restricts to a map ϕ_f on each fibre, and since we have

$$\phi \circ \mu = \mu' \circ (\phi \times \phi), \qquad \phi \circ \eta = \eta',$$

each ϕ_f is a group homomorphism $\mathcal{C}_f \longrightarrow \mathcal{C}'_f$. These provide the components of a natural transformation $\Gamma(\phi)$ between the associated natural systems. Recall that if $(D, \nu) = \Gamma((\mathcal{C}, p), \mu, \eta, (-)^{-1})$ is the natural system derived from $(\mathcal{C}, p)$, we have $D(u, v)(c) = \eta(u) \star_0 c \star_0 \eta(v)$. Then, since

$$\phi(\eta(u) \star_0 c \star_0 \eta(v) = \eta'(u) \star_0 \phi(c) \star_0 \eta'(v),$$

we have defined a natural transformation due the commutativity of

$$\begin{tikzcd}
D_f \arrow[r, "D(u,v)"] \arrow[d, "\Gamma(\phi)_f"'] & D_{ufv} \arrow[d, "\Gamma(\phi)_{ufv}"] \\
D'_f \arrow[r, "D'(u,v)"'] & D'_{ufv}
\end{tikzcd}$$

The last thing we need to check is compatibility with the composition pairings ν and ν', but as we saw earlier, this is automatic since $\Gamma(\phi)$ is a natural transformation of the natural systems.

The functoriality of the inverse construction involves routine checking. From a natural transformation $\psi : (D, \nu) \Rightarrow (D', \nu')$ respecting composition pairings, writing $((\mathcal{C}, p), \mu, \eta, (-)^{-1}) = \Lambda(D, \nu)$ and $((\mathcal{C}', p'), \mu, \eta, (-)^{-1}) = \Lambda(D', \nu')$, we get fibre maps $\Lambda(\psi)_f : C_f \to C'_f$, sending $(c, f) \in C_f$, where $c \in D_f$, to $(\psi_f(c), f)$. We use these maps to build a functor $\Lambda(\psi) : \mathcal{C} \to \mathcal{C}'$ between the categories, fibre by fibre. This is functorial, since composition is given by the composition pairings:

$$\begin{aligned}
\Lambda(\psi)((c, f) \star_0^{\mathcal{C}} (d, g)) &= \Lambda(\psi)(\nu_{f,g}(c, d), fg) \\
&= (\psi_{fg}(\nu_{f,g}(c, d), fg) \\
&= (\nu'_{f,g}(\psi_f(c), \psi_g(d)), fg) = (\psi_f(c), f) \star_0^{\mathcal{C}'} (\psi_g(d), f)
\end{aligned}$$

Furthermore, since it is defined fibre by fibre, it is a functor of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, *i.e.* we have $p = p' \circ \psi$.

We still need to see that ψ is a morphism of group objects, *i.e.* interacts correctly with the multiplication and identity maps. This is because each ψ_f is a group homomorphism $D_f \longrightarrow D'_f$ and the multiplication and identity maps are constructed from the multiplication and identity in each of those groups.

To conclude that we have an equivalence of categories we need to examine the composites $\Lambda\Gamma$ and $\Gamma\Lambda$.

Given a group object $((\mathcal{C},p),\mu,\eta,(-)^{-1})$, we have that $\Lambda\Gamma(\mathcal{C},p)$ consists of a category $\Lambda\Gamma(\mathcal{C})$ with objects those of $\mathcal{C}$, and for x and y such objects, hom-sets

$$\Lambda\Gamma(\mathcal{C})(x,y)=\coprod_{f\in\mathcal{B}(x,y)}\Gamma(\mathcal{C},p)_f=\coprod_{f\in\mathcal{B}(x,y)}C_f.$$

Furthermore, if $(c,d)\in\Lambda\Gamma(\mathcal{C})(x,y)$ and $(d,g)\in\Lambda\Gamma(\mathcal{C})(y,z)$, we have

$$\begin{aligned}(c,f)\star_0^{\Lambda\Gamma(\mathcal{C})}(d,g)&=(\mu(\Gamma(\mathcal{C},p)(1,g)(c),\Gamma(\mathcal{C},p)(f,1)(d)),fg)\\&=\mu(c\star_0^{\mathcal{C}}\eta(g),\eta(f)\star_0^{\mathcal{C}}d)\\&=\mu(c,\eta(f))\star_0^{\mathcal{C}}\mu(\eta(f),d)=c\star_0^{\mathcal{C}}d.\end{aligned}$$

The slight difference in results is cosmetic; it comes from the notation of elements in a coproduct. These compositions are in fact the same. Thus $\Lambda\Gamma(\mathcal{C})$ is equal to $\mathcal{C}$, and we also get the same functor p back via $\Lambda\Gamma$. Further routine checking shows that the multiplication and identity maps are carried over identically as well.

Now, given a natural system with composition pairing (D,ν), $\Gamma\Lambda(D)$ is the natural system which to an arrow f of $\mathcal{B}$ associates the fibre group $\Lambda(D,\nu)_f$, which is by definition equal to D_f. Additionally, writing $\Lambda(D,\nu)=((\mathcal{C},p),\mu,\eta,(-)^{-1})$, an extension (u,v) gives a group homomorphism sending $c\in D_f$ to $\Gamma\Lambda(D,\nu)(u,v)(c)=\eta(u)\star_0^{\mathcal{C}}c\star_0^{\mathcal{C}}\eta(v)$. Routine checking gives $\eta(u)\star_0^{\mathcal{C}}c\star_0^{\mathcal{C}}\eta(v)=D(u,v)(c)$ for all $c\in D_f$. Thus D and $\Gamma\Lambda(D)$ are the same natural system. We also easily verify that $\Gamma\Lambda(\nu)=\nu$.

These functors are thus mutually inverse, and the categories are isomorphic. □

Using the above results and Corollary 11.3.6, we also have:

11.4.4. Corollary ([77]). *There is an isomorphism of categories*

$$\mathbf{NatSys}(\mathcal{B},\mathbf{Ab})\simeq\mathbf{Ab}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}).$$

11.5. EXACT SEQUENCES OF ACTIONS

In this section, we recall notions from [62] concerning exactness in certain non-abelian categories. This is in the interest of first defining relative natural homotopy, and then showing that we obtain a long exact sequence of relative homotopy groups as in the classical case. Proposition 11.5.7 first appeared in [15] but is simply a special case of Theorem 6.4.9 in [62]. In what follows, we denote by **Gp** the category of groups and group homomorphisms, and by $\mathbf{Set}_*$ the category of pointed sets and pointed maps.

11.5.1. Semi-exact and homological categories. We would like to define an exactness property in the setting of non-abelian categories. To do this, we use a generalisation of kernels and cokernels.

An *ideal* N of a category $\mathcal{M}$ is a set of 1-cells of $\mathcal{M}$ stable by external composition, *i.e.* such that for all $g \in N$ and all f, h 1-cells of $\mathcal{M}$ for which it makes sense, we have $fgh \in N$. The elements of N are called *null morphisms*, and a 0-cell of $\mathcal{M}$ is called a *null object* when its identity is null.

We consider *closed* ideals; those in which every morphism in N factorises through an identity of N, or equivalently, through a null object. More precisely, an ideal N is closed if for every $f : A \to C \in N$, there exists a null object B and morphisms $g : A \to B$ and $h : B \to C$ such that $f = gh$. From now on, all ideals will be assumed to be closed.

The *kernel* with respect to a closed ideal N, or simply kernel when no confusion is possible, of a 1-cell $f : A \to B$ is a 1-cell $ker(f) : Ker(f) \to A$ annihilating f, *i.e.* such that $ker(f)f$ is null, and is minimal with this property in the sense that for all g such that gf is null, there exists a unique morphism $\overline{g}$ such that $g = \overline{g}ker(f)$.

Dually, the *cokernel* of f is a 1-cell $cok(f) : B \to Cok(f)$, such that $fcok(f)$ is null and every 1-cell annihilating f on the right factorises uniquely through $cok(f)$. The situation is described diagrammatically below:

$$\begin{tikzcd}
Ker(f) \arrow[r, "ker(f)"] & A \arrow[r, "f"] & B \arrow[r, "cok(f)"] \arrow[rd, "g'"'] & Cok(f) \arrow[d, dashed, "\overline{g'}"] \\
C \arrow[u, dashed, "\overline{g}"] \arrow[ru, "g"'] & & & C'
\end{tikzcd}.$$

Note that the kernel and cokernel of a 1-cell, if they exist, are uniquely determined. When a 1-cell f of $\mathcal{M}$ is the kernel of another, we say that f is a *normal mono*, whereas when it is a cokernel, we call it a *normal epi*.

A pair $(\mathcal{M}, N)$ define a *semi-exact category* when N is a closed ideal of $\mathcal{M}$, and every 1-cell of $\mathcal{M}$ has a kernel and cokernel with respect to N. In such a category, we additionally define the (normal) *image* of a morphism f by setting $im(f) = ker(cok(f))$, and dually, the (normal) *coimage* $coim(f) = cok(ker(f))$. We say that a morphism f is *exact* when it factorises $f = qn$ with q a normal epi and n a normal mono.

11.5.2. Example. In the category **Gp**, (co-)kernels and (co-)images coincide with the usual notions. Let us examine the case of $\mathbf{Set}_*$. This category is *pointed*, meaning that its initial and terminal objects coincide; we call it the *zero object*. Such categories come with a canonical structure of semi-exact category; we take null morphisms to be those which factorise through the zero object. Thus, a 1-cell $f : (X, x) \to (Y, y)$ of $\mathbf{Set}_*$ is null if and only if $f(X) = \{y\}$. Its kernel is the inclusion $ker(f) : (f^{-1}(\{y\}), x \to (X, x)$, while its cokernel is the quotient $cok(f) : (Y, y) \to (Y/f(X), [y])$, where $[y]$ is the class of the base point y in the quotient space $Y/f(X)$.

11.5.3. Exact sequences. In a semi-exact category $\mathcal{M} = (\mathcal{M}, N)$, we can generalise the properties of sequences of morphisms. Given composable morphisms f and g of $\mathcal{M}$, we say that the sequence

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

- is *of order two* (at B) when fg is null.
- is *exact* (at B) when $im(f) = ker(g)$.
- is *short exact* (at B) when $f = ker(g)$ and $g = cok(f)$.

As in the classic setting, a short exact sequence is exact, and an exact sequence is of order two.

11.5.4. The category of actions. Here we recall the category **Act** of actions of groups on pointed sets from [62]. This category will be of use because we can view **Gp** and $\mathbf{Set}_*$ in **Act** while preserving exactness properties.

An *action* is a pair (X, G) where X is a pointed space, whose base point we shall denote by 0_X, and G is a group with identity element 1_G, equipped with a right action of G on X. The base point of X is not assumed to be fixed by the action, and we will write

$$G_0 = Fix_G(0_X) = \{g \in G \mid 0_X \cdot g = 0_X\}$$

to denote the subset of G fixing the base point.

In **Act**, a morphism is a pair $f = (f', f'') : (X, G) \to (Y, H)$ where $f' : X \to Y$ is a morphism of pointed sets, and $f'' : G \to H$ is a morphism of groups compatible with the action in the sense that for all $g \in G$ and all $x \in X$,

$$f'(x \cdot g) = f'(x) \cdot f''(g).$$

Abusing notation, we will denote f' and f'' by f when no confusion is possible.

Now we define a closed ideal of morphisms which makes **Act** a semi-exact category as introduced by Grandis in [62, Section 6.4]. This will be inherited from the null ideal described above in the case of $\mathbf{Set}_*$; we take the null morphisms to be those $f = (f', f'')$ such that f' is a null morphism of $\mathbf{Set}_*$, *i.e.* $f'(X) = \{0_Y\}$. This ideal, which we will denote N, is closed, since null objects are actions on the trivial pointed set, and a null morphism $f : (X, G) \to (Y, H)$ clearly factorises through $(\{0_Y\}, H_0)$.

To see that **Act** is semi-exact, we explicit the kernel and cokernel of a null morphism $f : (X, G) \to (Y, H)$:

i) Its kernel is the inclusion

$$(Ker(f'), f^{-1}(H_0)) \longrightarrow (X, G).$$

Recall that $Ker(f') = f^{-1}(\{0_Y\})$, and let it be observed that $f^{-1}(H_0)$ is the subset of G consisting of elements g such that $x = x' \cdot g$ for some $x, x' \in Ker(f')$.

ii) Its cokernel is the projection

$$(Y, H) \longrightarrow (Y/R, H)$$

where R is an equivalence relation on Y generated by identifying all of the points of $f(X)$ in a way which is compatible with the action of H. More precisely, $y \equiv_R y'$ if and only if either y or y' is an element of $f(X)$ and there exists some $h \in H$ with $y = y' \cdot h$.

This shows that every morphism of **Act** has a kernel and cokernel with respect to N, *i.e.* that $(\mathbf{Act}, N)$ is a semi-exact category. In fact, a little extra work allows us to show that it is even *homological*, meaning that normal monos and normal epis are stable under composition, and that whenever m is a normal mono and q a normal epi such that $ker(q)$ factorises through m, mq is exact.

11.5.5. Embeddings of Gp and Set$_*$ in Act. There exist embeddings of the categories **Gp** and **Set**$_*$ into the category **Act** that preserve exactness of sequences and morphisms, see again [62]. In the case of **Set**$_*$, there are adjoint functors,

$$J : \mathbf{Set}_* \to \mathbf{Act}, \qquad V : \mathbf{Act} \to \mathbf{Set}_*,$$

defined by $J(X) = (X, \{1\})$ and $V(X, G) = X/G$ for all pointed sets X and groups G with a right action on X, where $(X, \{1\})$ is the action of the trivial group on X, and X/G is the quotient of X by the G-orbits of the action, pointed at the class of 0_X. The functor J induces an equivalence of categories between **Set**$_*$ and the full homological subcategory of **Act** consisting of actions of the trivial group. This, along with the fact that J preserves null morphisms, means that it preserves exactness of sequences.

On the other hand, the category **Gp** can be realised as a retract of the category **Act**, via the functors

$$K : \mathbf{Gp} \to \mathbf{Act}, \qquad R : \mathbf{Act} \to \mathbf{Gp},$$

defined by $K(G) = (|G|, G)$ and $R(X, G) = G/\overline{G_0}$, where $(|G|, G)$ is the usual right action of G on the underlying set $|G|$, pointed at 1_G. Recall that this action is transitive. In the definition of R, $G/\overline{G_0}$ is the quotient of G by the invariant closure in G of the subgroup G_0 stabilising the base point 0_X of X. These show that **Gp** is a retract of **Act** in the sense that $R \circ K = id_{\mathbf{Gp}}$ since the action of G on itself is transitive. As a consequence, a sequence of groups viewed in **Act** is exact if and only if the sequence is exact in the usual sense.

11.5.6. Relative homotopy sequences in Act. Here we briefly recall the definition of relative homotopy groups in the usual setting of pairs of topological spaces, and then see that these fit into an exact sequence in **Act**.

Let us recall the definition of the relative homotopy groups of a pair of topological spaces. For $n \geq 1$, let I^n denote the n-dimensional unit cube $[0,1]^n$. We single out the face $I^{n-1} := \{(x_1, \ldots, x_n) \mid x_n = 0\}$, and define J^n to be the closure of $\partial I^n \setminus I^{n-1}$ in I^n. Given a *pointed pair* of topological spaces (X, A), *i.e.* a space X and a subspace

$A \subseteq X$ pointed at $x \in A$, we define, for $n \geq 1$, the n^{th} *relative homotopy* of (X, A) by setting

$$\pi_n(X, A) := [f : (I^n, \partial I^n, J^n) \to (X, A, x)]$$

i.e. the homotopy classes of maps $f : I^n \to X$ with $f(\partial I^n) \subseteq A$ and $f(J^n) = \{x\}$. The homotopies between such maps must satisfy the same conditions.

Note that for $n = 1$, this is not a group for concatenation. Indeed, a map $f : (I, \{0, 1\}, 1) \to (X, A, x)$ is required to end at x, but can start anywhere in A, and therefore such maps can in general not be concatenated. We consider $\pi_1(X, A)$ as a pointed set, the pointed element being the class of paths f such that f is homotopic to a path g with its image contained in A, *i.e.* $g([0, 1]) \subseteq A$. For $n \geq 2$, $\pi_n(X, A)$ forms a group under concatenation, and is abelian for $n \geq 3$. For $f : I^n \to X$, its class in $\pi_n(X, A)$ is the identity element if, and only if, it is homotopic to a map $g : I^n \to X$ with its image contained in A. We refer the reader to [70] for more information about relative homotopy.

The assignment of relative homotopy groups to a pointed pair of spaces is functorial. Its domain is the category of pointed pairs of topological spaces, denoted by $\top_{*2}$, in which a morphism $f : (X, A, x) \to (Y, B, y)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq f(B)$ and $f(x) = y$, and its codomain is $\mathbf{Set}_*$ for $n = 1$, $\mathbf{Gp}$ for $n = 2$ and $\mathbf{Ab}$ for $n \geq 3$. We can therefore consider these as functors with values in $\mathbf{Act}$ for all $n \geq 1$.

11.5.7. Proposition ([15]). *Given a pointed pair of topological spaces* (X, A), *we get an exact sequence in* $\mathbf{Act}$*:*

$$\ldots \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \xrightarrow{\partial_n} \pi_{n-1}(A) \to \ldots$$
$$\ldots \xrightarrow{v} \pi_1(X) \xrightarrow{f} (\pi_1(X, A), \pi_1(X)) \xrightarrow{g} \pi_0(A) \xrightarrow{h} \pi_0(X) \to \pi_0(X, A) \to 0.$$

Below we outline why this sequence is exact in the particular case of relative homotopy; as stated in [62], it is a consequence of Theorem 6.4.9 from the same reference. Note that all of the arrows are induced by inclusions, except the last non-trivial arrow and the maps ∂_n, which are given by restriction to the distinguished face: $\partial_n([\sigma]) = [\sigma]_{|I^{n-1}}$. Also recall that we have not defined a relative homotopy group for $n = 0$; the object $\pi_0(X, A)$ is defined to be $Cok(h)$. It is thus the quotient of the set of path-connected components of X obtained by identifying the components which intersect A.

All the terms above $\pi_1(X)$ are groups, and the existence of this long exact sequence in $\mathbf{Gp}$ is well known. It is in fact a special case of a Puppe sequence, see [102, Theorem 11.39]. Since exactness is carried into $\mathbf{Act}$, we know that the sequence is exact in $\mathbf{Act}$ up to this object.

As observed above, $\pi_1(X, A)$ is not a group, but a pointed set. The group $\pi_1(X)$ acts on it by concatenation; the elements of $\pi_1(X, A)$ have starting point 0_X, and so we can concatenate with elements of $\pi_1(X)$ at that end. The sequence is exact at $\pi_1(X) =$

$(|\pi_1(X)|, \pi_1(X))$ since the image of v is precisely $Ker(f')$; indeed, $Fix_{\pi_1(X)}(0_{\pi_1(X,A)}) = \pi_1(A)$.

The map g in the above sequence sends $(\tau, \sigma) \in (\pi_1(X, A), \pi_1(X))$ to the path-connected component of $\tau(0) \in A$. We therefore view it is a pointed set map from $\pi_1(X, A)$ to $\pi_0(A)$. The sequence is exact at $(\pi_1(X, A), \pi_1(X))$ because the antecedents under g of the pointed element of $\pi_0(A)$, namely the component containing the base point x, are elements of the orbit of the pointed element 0 of $\pi_1(X, A)$, *i.e.* $0 \cdot \pi_1(X)$. This coincides with the image of f since it is defined by sending $\sigma \in \pi_1(X)$ to $(0 \cdot \sigma, \sigma)$.

Lastly, we show exactness at $\pi_0(A)$, since at $\pi_0(X)$ it follows by definition of $\pi_0(X, A)$. Observe that the inverse image under h of the pointed element $[x]$ in $\pi_0(X)$ is exactly $\{[x]\}$, the pointed element in $\pi_0(A)$, since h is induced by the inclusion $A \hookrightarrow X$. Furthermore, for $\tau \in \pi_1(X, A)$, $g(\tau)$ is necessarily in the same path connected component as x. Thus, the image of g coincides with the kernel of h.

11.6. DIRECTED HOMOLOGY AND HOMOTOPY

In this section we recall the notion of directed spaces from [61], and define algebraic invariants for these spaces, natural homotopy and natural homology, as introduced in [31, 32].

11.6.1. Directed spaces. Recall from [61] that a *directed space*, or *dispace*, is a pair $\mathcal{X} = (X, dX)$, where X is a topological space and dX is a set of paths in X, *i.e.* continuous maps from $[0, 1]$ to X, called *directed paths*, or *dipaths* for short, satisfying the three following conditions:

i) Every constant path is directed,

ii) dX is closed under monotonic reparametrization,

iii) dX is closed under concatenation.

We will denote by $f \star g$ the concatenation of dipaths f and g, defined via monotonic reparametrization. A morphism $\varphi : (X, dX) \to (Y, dY)$ of dispaces is a continuous function $\varphi : X \to Y$ that preserves directed paths, *i.e.* , for every path $p : [0, 1] \to X$ in dX, the path $\varphi_* p : [0, 1] \to Y$ belongs to dY. The category of dispaces is denoted **dTop**. An isomorphism in **dTop** from (X, dX) to (Y, dY) is a homeomorphism from X to Y that induces a bijection between the sets dX and dY.

Note that the forgetful functor U : **dTop** → **Top** admits left and right adjoint functors. The left adjoint functor sends a topological space X to the dispace (X, X_d), where X_d is the set of constant directed paths. The right adjoint sends X to the dispace $(X, X^{[0,1]})$, where $X^{[0,1]}$ is the set of all paths in X.

For a dispace $\mathcal{X} = (X, dX)$ and x,y in X, we denote by $\overrightarrow{Di}(\mathcal{X})(x, y)$ the space of dipaths f in X with source $x = f(0)$ and target $y = f(1)$, equipped with the compact-open topology.

11.6.2. The trace category. The *trace space* of a dispace $\mathcal{X}$ from x to y, denoted by $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$, is the quotient of $\overrightarrow{Di}(\mathcal{X})(x,y)$ by monotonic reparametrization, equipped with the quotient topology. The *trace* of a dipath f in $\mathcal{X}$, denoted by $\overline{f}$ or f if no confusion is possible, is the equivalence class of f modulo monotonic reparametrization. The concatenation of dipaths of $\mathcal{X}$ is compatible with this quotient, inducing a concatenation of traces defined by $\overline{f} \star \overline{g} := \overline{f \star g}$, for all dipaths f and g of $\mathcal{X}$. We will denote by

$$\overrightarrow{\mathbf{P}} : \mathbf{dTop} \to \mathbf{Cat}$$

the functor which associates to a dispace $\mathcal{X}$ the *trace category of* $\mathcal{X}$, whose 0-cells are points of X, 1-cells are traces of $\mathcal{X}$, and composition is given by concatenation of traces.

11.6.3. Trace diagrams. The *pointed trace diagram in* $\mathbf{Top}_*$ of a dispace $\mathcal{X}$ is the functor

$$\overrightarrow{T}_*(\mathcal{X}) : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathbf{Top}_*$$

sending a trace $\overline{f} : x \to y$ to the pointed topological space $(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), \overline{f})$, and a 1-cell $(\overline{u}, \overline{v})$ of $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ to the continuous map

$$\overline{u} \star \_ \star \overline{v} : \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{X})(x',y')$$

which sends a trace $\overline{f}$ to $\overline{u} \star \overline{f} \star \overline{v}$. The functor $\overrightarrow{T}_*(\mathcal{X})$ extends to a functor

$$\overrightarrow{T}_* : \mathbf{dTop} \to \mathbf{opNat}(\mathbf{Top}_*)$$

sending a dispace $\mathcal{X}$ to the pair $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{T}_*(\mathcal{X}))$. Observe that a morphism of dispaces $\varphi : \mathcal{X} \to \mathcal{Y}$ induces continuous maps

$$\varphi_{x,y} : \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{Y})(\varphi(x), \varphi(y))$$

for all points x, y of X. Thus we obtain natural transformations between the corresponding trace diagrams:

$$\overrightarrow{\varphi}_* : \overrightarrow{T}_*(\mathcal{X}) \Rightarrow \overrightarrow{T}_*(\mathcal{Y}).$$

11.6.4. Natural homotopy and natural homology. Recall from [31] that the 1^{st} *natural homotopy functor of* $\mathcal{X}$ is the natural system denoted by $\overrightarrow{P}_1(\mathcal{X}) : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathbf{Set}$, and defined as the composite

$$F\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\overrightarrow{T}_*(\mathcal{X})} \mathbf{Top}_* \xrightarrow{\pi_0} \mathbf{Set}_*,$$

where π_0 is the 0^{th} homotopy functor with values in $\mathbf{Set}_*$. That is, for a trace $\overline{f}$ on $\mathcal{X}$ from x to y,

$$\overrightarrow{P}_1(\mathcal{X})_{\overline{f}} = (\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)), [\overline{f}]),$$

where $[\overline{f}]$ denotes the path-connected component of $\overline{f}$ in $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$. For $n \geq 2$, the n^{th} *natural homotopy functor of* $\mathcal{X}$, denoted by $\overrightarrow{P}_n(\mathcal{X}) : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathbf{Gp}$, is defined as the composite

$$F\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\overrightarrow{T}_*(\mathcal{X})} \mathbf{Top}_* \xrightarrow{\pi_{n-1}} \mathbf{Gp},$$

where π_{n-1} is the $(n-1)^{th}$ homotopy functor. Note that for $n \geq 3$, the functor $\overrightarrow{P}_n(\mathcal{X})$ has values in $\mathbf{Ab}$. Finally, for $n = 0$, we define $\overrightarrow{P}_0(\mathcal{X}) : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathbf{Set}_*$ as the functor sending a trace $\overline{f}$ to the pointed singleton $(\{\overline{f}\}, \overline{f})$.

Using the inclusion functors $J : \mathbf{Set}_* \to \mathbf{Act}$ and $K : \mathbf{Gp} \to \mathbf{Act}$ defined in Section 11.5.5, the classical homotopy functors can be realised as functors $\pi_n : \mathbf{Top}_* \to \mathbf{Act}$, for all $n \geq 0$. With this interpretation, natural homotopy can be resumed by functors

$$\overrightarrow{P}_n(\mathcal{X}) : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathbf{Act},$$

for all $n \geq 0$.

Recall from [32], that for $n \geq 1$, the n^{th} *natural homology functor of* $\mathcal{X}$ is the functor denoted by $\overrightarrow{H}_n(\mathcal{X}) : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathbf{Ab}$, and defined as the composite

$$F\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\overrightarrow{T}(\mathcal{X})} \top \xrightarrow{H_{n-1}} \mathbf{Ab}$$

where H_{n-1} is the $(n-1)^{th}$ singular homology functor.

The functors $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$, for $\mathcal{X}$ in $\mathbf{dTop}$, extend to functors

$$\overrightarrow{P}_n : \mathbf{dTop} \longrightarrow \mathbf{opNat}(\mathbf{Act}), \quad \text{and} \quad \overrightarrow{H}_n : \mathbf{dTop} \longrightarrow \mathbf{opNat}(\mathbf{Ab}),$$

sending a dispace $\mathcal{X}$ to $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{P}_n(\mathcal{X}))$ and $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{H}_n(\mathcal{X}))$ respectively.

11.6.5. Proposition. *Given a topological space* X*, the dispace* $\mathcal{X} = (X, X^{[0,1]})$ *is such that for every* x *in* X*,*

$$\overrightarrow{P}_n(\mathcal{X})_{c_x} \cong \pi_n(X, x),$$

where c_x *denotes the trace of the constant dipath equal to* x*.*

Proof. Recall that for any $x \in X$, the loop space $\Omega(X, x)$ is the set of all continuous paths $p : \mathbb{S}^1 \to X$ given the compact-open topology, and is thus homeomorphic to $\overrightarrow{Di}(\mathcal{X})(x, x)$. As a consequence of Eckmann-Hilton duality, for any topological space X and any $n \geq 1$,

$$\pi_n(X, x) \cong \pi_{n-1}(\Omega(X, x)).$$

The quotient of $\Omega(X, x)$ by monotonic reparametrization is the space $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x, x)$, and since paths in the same reparametrization class are homotopic, we have $\overrightarrow{P}_n(\mathcal{X})_{c_x} \cong \pi_n(X, x)$. □

As a consequence, given a dispace $\mathcal{X} = (X, X^{[0,1]})$, if X is n-connected, then for every $x \in X$, the space $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x, x)$ is also $(n-1)$-connected. Applying the Hurewicz theorem, Proposition 11.6.5 yields the following result.

11.6.6. Corollary. *For $n \geq 1$ an $(n-1)$-connected topological space X, the dispace $\mathcal{X} = (X, X^{[0,1]})$ is such that for every x in X*

$$\overrightarrow{H}_i(\mathcal{X})_{c_x} \cong H_i(X),$$

for all $i \leq n$.

11.7. PERSISTENT HOMOLOGY

This section deals with basic notions of persistent homology in order to fix the notations and make the thesis self-contained. We recall definitions of persistence objects and persistence homology, and the classification in terms of barcodes. We recall also an algorithm for computing persistent homology from [21]. We refer the reader to [19, 35, 36] for complete accounts of persistent homology.

11.7.1. Persistence complexes. Given a poset P, considered as a category, a *P-persistence object in a category $\mathcal{C}$* is a functor $\Phi : P \to \mathcal{C}$. Explicitly, it is given by a collection $\{C_x\}_x$ of objects in $\mathcal{C}$ indexed by the elements of P, and such that for all $x \leq y$ in P, there exists a unique map $\phi_{x,y} : C_x \to C_y$, such that with $\phi_{y,z} \circ \phi_{x,y} = \phi_{x,z}$ whenever $x \leq y \leq z$. We denote by $P_{pers}(\mathcal{C})$ the functor category of P-persistence objects in $\mathcal{C}$. When $\mathcal{C}$ is the category of simplicial complexes, chain complexes, groups... the objects of $P_{pers}(\mathcal{C})$ are called P-persistence simplicial complexes, chain complexes, groups...

In particular, considering the poset $\mathbb{N}$ of natural numbers with the usual order, a *positive $\mathbb{N}$-persistence complex*, or *persistence complex* for short, over a ground ring R is a family of chain complexes $C = \{C^i_*\}_{i \geq 0}$ over R, together with chain maps $f^i : C^i_* \to C^{i+1}_*$, giving, for every $k \in \mathbb{N}$, the following diagram in the category of R-modules:

$$\begin{tikzcd} C^0_k \arrow[r, "f^0_k"] & C^1_k \arrow[r, "f^1_k"] & \dots \arrow[r] & C^i_k \arrow[r, "f^i_k"] & C^{i+1}_k \arrow[r] & \dots \end{tikzcd}$$

A *persistence module* M is a persistence complex concentrated in degree zero, *i.e.* a family of R-modules $\{M^i\}_{i \geq 0}$, together with maps $f^i : M^i \to M^{i+1}$.

The persistence complex C is called of *finite type* if each R-module C^i_k is finitely generated, and if there exists some N such that the maps f^i are isomorphisms for $i \geq N$.

11.7.2. Persistent homology. Recall that a *simplical complex* is a set K, together with a collection $\mathcal{K}$ of subsets of K, satisfying the following two conditions:

ii) for every $v \in K$, $\{v\} \in \mathcal{K}$, and $\{v\}$ is called a *vertice* of K;

iiii) $\alpha \in \mathcal{K}$ and $\beta \subseteq \alpha$ implies $\beta \in \mathcal{K}$.

A *k-simplex of K* is an element σ of $\mathcal{K}$ whose cardinal $|\sigma|$ is equal to $k+1$. An *orientation* of a k-simplex $\sigma = \{v_0, \dots, v_k\}$ is an equivalence of orderings of the v_i in σ; two orderings are equivalent if they can be obtained from an even permutation. A simplex with an orientation is called an oriented simplex, and we write $[v_0, \dots, v_k]$ or $[\sigma]$ to denote the equivalence class.

Denote by $C_k(K)$ the k^{th} *chain module* of K defined as the free R-module on oriented k-simplices of K. The boundary operator $\partial_k : C_k(K) \to C_{k-1}(K)$ is the map defined on any simplex $\sigma = \{v_0, \dots, v_k\}$ by setting

$$\partial_k(\sigma) = \sum_i (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_k],$$

where in the right side $\widehat{v_i}$ indicates that the vertex v_i is eliminated from the simplex. Denote by $Z_k(K) = \ker \partial_k$, $B_k(K) = \operatorname{Im} \partial_{k+1}$, and $H_k(K) = Z_k(K)/B_k(K)$ the *cycle*, *boundary*, and *homology* modules respectively.

A *subcomplex* of K is a simplicial complex L such that $L \subseteq K$. A *filtered complex* is a complex K together with a *filtration*, that is a nested sequence of subcomplexes:

$$K^0 \subseteq K^1 \subseteq K^2 \subseteq \ldots \subseteq K^n = K.$$

Given such a filtration, we recall the definition of the *p-persistent kth homology group of* K^i, given as the quotient

$$H_k^{i,p}(K) := Z_k(K^i)/B_k(K^{i+p}) \cap Z_k(K^i).$$

All of these groups can be combined in a single persistence module. Indeed, we consider the persistence complex $C(K) = \{C_*(K^i)\}_{i \in \mathbb{N}}$, in which the chain maps $f^i : C_*(K^i) \to C_*(K^{i+1})$ are induced by the inclusions $K^i \to K^{i+1}$. Applying the k^{th} homology functor H_k to each complex, we obtain $H_k(C(K)) := \{H_k\left(C(K_i)_*\right)\}_{i \in \mathbb{N}}$, which has the structure of persistence module over the ring with which we take coefficients. Furthermore, we recover the notion of p-persistent k^{th} homology group of K^i. Indeed, denoting by $\eta_k^{i,p} : H_k(K^i) \to H_k(K^{i+p})$ the map induced by the inclusion $K^i \to K^{i+p}$, we have $im(\eta_k^{i,p}) \simeq H_k^{i,p}(K)$ [37].

If K is finite, the persistence complex $C(K)$ is of finite type, and thus the homology $H_k(C(K))$ is of finite type.

11.7.3. Classification of persistence modules. Given a persistence R-module $(M^i, \phi_{i,i+1})_i$, we define a graded module over $R[t]$ by setting

$$\alpha(M) = \bigoplus_{i=0}^{\infty} M^i,$$

where the R-module structure is given by the direct sum of the structures of the components, and where the action of t is given by the $\phi_{i,i+1}$, *i.e.*

$$t \cdot (m_0, m_1, \dots, m_n, \dots) = (0, \phi_{0,1}(m_0), \phi_{1,2}(m_1), \dots, \phi_{n,n+1}(m_n), \dots).$$

This correspondence establishes an equivalence between the category of persistence R-modules of finite type and the category of finitely generated graded $R[t]$-modules [38].

Combining this with the characterisation of finitely generated graded modules, we know that a persistence module with coefficients in some field $\mathbb{K}$ corresponds to a decomposition akin to (11.7.1) below.

We fix some filtration $(K_i)_{i\in\mathbb{N}}$ of a finite complex K. We will be considering finitely generated (tame) graded modules over the graded ring $\mathbb{K}[t]$, where $\mathbb{K}$ is some ground field. The ring $\mathbb{K}[t]$ is a principal ideal domain, and thus we have, as a result of a classification theorem, see [33], that a finitely generated non-negatively graded $\mathbb{K}[t]$-module is isomorphic to

$$\left(\bigoplus_{i=1}^{n}\Sigma^{k_i}\mathbb{K}[t]\right)\oplus\left(\bigoplus_{j=1}^{m}\Sigma^{l_j}\mathbb{K}[t]/(t^{h_j})\right) \tag{11.7.1}$$

for some n, m and families of natural numbers $(k_i)_i$, $(l_j)_j$ and $(h_j)_j$, where Σ^k denotes a k-shift in grading.

11.7.4. Barcodes. Using the classification theorem for finitely generated generated $\mathbb{K}[t]$-modules above, we know that given a $\mathbb{N}$-persistence $\mathbb{K}$-vector space $\{V_k, \psi_{k,k+1}\}_k$, *i.e.* a persistence $\mathbb{K}$-module, we can find $m_i \in \mathbb{N}$ and $n_i \in \mathbb{N}\cup\{\infty\}$ such that

$$\{V_n\}_n \simeq \bigoplus_{i=0}^{N} U(m_i, n_i),$$

where for $m \leq n$, $U(m,n)$ is the $\mathbb{N}$-persistence $\mathbb{K}$-vector space such that $U(m,n)_t = \{0\}$ for $t < m$ and $t > n$ and $U(m,n)_t = \mathbb{K}$ otherwise. The associated maps $\psi_{s,t}$ are given by the identity whenever $m \leq t \leq n$. This decomposition is unique in the sense that the collection of pairs $\{(m_i, n_i)_i\ \}$ is unique up to the order of the factors. This collection of pairs is called the *barcode* associated to $\{V_n\}_n$. The barcode classifies tame $\mathbb{N}$-persistence F-vector spaces just as dimension classifies finite dimensional vector spaces.

In summary, calculating the persistent homology of a simplicial complex K with respect to a filtration $(K^i)_i$ thereof goes as follows: First, we obtain a persistence complex $C(K) = \{C_*(K^i)\}_{i\in\mathbb{N}}$ to which we apply the homology functor. Taking coefficients in a field $\mathbb{K}$, this then gives $H_k(C(K)) := \{H_k\left(C(K_i)_*\right)\}_{i\in\mathbb{N}}$, a persistence $\mathbb{K}$-vector space. Finally, from this we calculate a barcode, as described in the next section.

11.7.5. Algorithm for computing persistent homology. Throughout this section, we denote by $\{e_j\}$ and $\{\hat{e}_i\}$ to represent homogenous bases for the persistence F-modules C_k and Z_{k-1}. Denote by M_k the matrix of ∂_k in these bases. The usual procedure for calculating homology is to reduce the matrix to Smith normal form and read off the description of H_k from the diagonal elements. We compute these bases and matrix representations by induction on k. For $k = 1$, the standard basis of $C_0 = Z_0$ is homogenous and we may proceed as usual.

Suppose now that we have a representation M_k of ∂_k relative to the standard basis $\{e_j\}$ of C_k and a homogeneous basis $\{\hat{e}_i\}$ of Z_{k-1}. For induction, we must compute a homogeneous basis for Z_k and represent ∂_{k+1} relative to the computed basis for Z_k and the standard basis of C_{k+1}.

We begin by sorting the basis $\{\hat{e}_i\}$ in reverse degree order and then transform the matrix M_k into *column-echelon form* $\tilde{M}_k$. This is a lower staircase form, the general form of which is depicted below, in which each landing is of width one, the steps have variable height. A *pivot* is the first non-zero value in a column (the boxed entries in the figure) and a row (resp. column) with a pivot is called a *pivot row* (resp. *column*).

$$\left(\begin{array}{cccccc} \boxed{*} & 0 & \cdots & & & 0 \\ & \boxed{*} & 0 & \cdots & & \vdots \\ & * & 0 & \cdots & & \\ & & \boxed{*} & 0 & \cdots & \end{array}\right)$$

A result from [120] tells us that the diagonal elements in Smith normal form are the same as the pivots in column-echelon form and that the degrees of the corresponding basis elements are also the same in both cases. This gives us the following result:

11.7.6. Proposition ([120]). *Let $\mathcal{C}_*$ be a persistence chain complex and let $\tilde{M}_k$ be the column-echelon form for ∂_k relative to (homogeneous) bases $\{e_j\}$ and $\{\hat{e}_i\}$ for $\mathcal{C}_k$ and Z_{k-1} respectively. Each row contributes to the persistent homology H_{k-1} of $\mathcal{C}_*$ in the following way:*

- *If row i is a pivot row with pivot t^n, then it contributes $\Sigma^{\deg \hat{e}_i} F[t]/t^n$.*
- *If row i is not a pivot row, it contributes $\Sigma^{\deg \hat{e}_i} F[t]$.*

where we recall that these contributions correspond to factors in the characterisation (11.7.1).

Notice that in calculating the column-echelon form, we have only switched column positions, or replaced a column by linear combinations of others, in such a way as to preserve homogeneity and degree of corresponding basis elements. Therefore, a homogeneous basis of Z_k is given by the non-pivot (*i.e.* zero) columns in the column-echelon form.

Now, if M_{k+1} is the matrix of ∂_{k+1} in the standard bases of C_{k+1} and C_k. We know that $M_k Mk + 1 = 0$, and that this is unchanged by elementary operations. The operations on columns of M_k correspond to operations on rows in M_{k+1}. These operations zero out rows in M_{k+1} which correspond to pivot columns in M_k and give us a representation of M_k relative to the standard basis of C_{k+1} and the basis we have computed for Z_k. This gives us the following result:

11.7.7. Lemma ([120]). *To represent ∂_{k+1} relative to the standard basis of C_{k+1} and the computed basis for Z_k, simply delete rows in the standard presentation of ∂_{k+1} which correspond to pivot columns in $\tilde{M}_k$.*

This concludes the inductive argument, and provides an algorithm for computing persistent homology.

CHAPTER 12.
NATURAL HOMOTOPY

We consider the directed space $\mathcal{X} = (X, dX)$, pictured in Figure 12.1, and invert the time flow. If we orient the time flow from left to right and from bottom to top, we need to rotate its representation as a dispace, as shown right of Figure 12.1. The concurrent processes modelled by these two dispaces should not be considered as the same under any form of well accepted equivalence. These two concurrent programs actually have equivalent prime event structure representations, see [59], that are not bisimulation equivalent [114].

Fajstrup and Hess noted that natural homotopy and homology theories do not distinguish between these two cases, but produce isomorphic natural systems [87]. We will show that this problem is solved using the notion of composition pairing recalled in Chapter 11, in particular see Section 11.3.

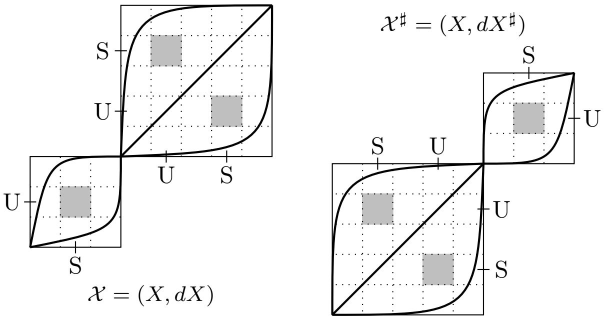


Figure 12.1: The dispace of a concurrent program and its time-reversed dispace.

First, in Section 12.1, we apply the results from Section 11.1 and 11.2 to the natural systems defined by the natural homotopy and homology functors. There are essentially three cases; for $n \geq 3$, where we have natural systems of abelian groups, and for $n = 2$, we obtain natural systems of groups, so we have check the commutator condition and find the associated composition pairing. These two cases will be treated together, resulting in Theorem 12.1.4, whereas the case for $n = 1$ is handled in Theorem 12.1.3 after defining the notion of split object in Section 12.1.1. Putting together these theorems, as well as the correspondence recalled in Section 11.4 from [93], we obtain Theorem 12.1.5, relating natural homotopy to group objects in a fixed object slice category. We describe the structure of these objects for natural homotopy and homology in Sections 12.1.6 and 12.1.7, respectively. Finally, in Section 12.1.8, we discuss the fundamental category of a directed space obstructions for defining natural homotopy and homology on connected components of path spaces.

In Section 12.2 we begin by formally defining the time-reversal of a dispace in Section 12.2.1, and then in Section 12.2.2 we define the notion of (strong) time-reversal of functors **dTop** $\to$ **Cat** with respect to opposition in **Cat**. We then show that without composition pairings, the natural homotopy and homology functors associated to a dispace do not detect time-reversal. This time-symmetry of the original invariants is the subject of Section 12.2.3. Section 12.2.4 contains the main theorems of this chapter, namely Theorems 12.2.5 and 12.2.6. These express that when equipped with a composition pairing, the invariants capture are strongly time-reversal. We conclude Section 12.2 defining a notion of time-reversal relative to the category $\mathbf{opNat}(\mathbf{Act})$ of natural systems of actions and proving Theorem 12.2.8, which states that time-reversal of a functor **dTop** $\to$ **Cat** is equivalent to the notion for $\mathbf{opNat}(\mathbf{Act})$.

Finally in Section 12.3, we focus on further enriching natural homotopy by defining a notion of relative natural homotopy. In particular, we prove Theorem 12.3.2, which states that a long exact sequence of homotopy groups may be constructed from a pair $(\mathcal{X}, \mathcal{A})$ of dispaces. We apply this to the special case of fibrations, resulting in Theorem 12.3.5.

Unless otherwise stated, definitions and results in this chapter are original contributions, first published in [15].

12.1. DIRECTED HOMOTOPY AS AN INTERNAL GROUP OR A SPLIT OBJECT

In this section we show that for any dispace $\mathcal{X}$, the natural systems $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$ admit composition pairings. We treat the case $\overrightarrow{P}_1(\mathcal{X})$ in Theorem 12.1.3 separately from the the case $\overrightarrow{P}_n(\mathcal{X})$ for $n \geq 2$ in Theorem 12.1.4. Finally, using the equivalence of categories stated in Theorem 11.4.3 and what is explained below in Section 12.1.1, we describe the natural homotopy functor $\overrightarrow{P}_n(\mathcal{X})$ as split or group objects in the category $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. We also treat the case of the natural homology functors $\overrightarrow{H}_n(\mathcal{X})$, for $n \geq 1$, which we describe as internal abelian groups in the category $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$.

12.1.1. Natural systems and split objects. Given a category $\mathcal{B}$, we define *the category of split objects in* $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, denoted by $\mathbf{Split}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B})$, as the full subcategory of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ whose objects are pairs $((\mathcal{C}, p), \epsilon)$, where $(\mathcal{C}, p)$ is an object of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ and

ϵ is a morphism of $\mathbf{Cat}_{\mathcal{B}_0}$ such that the following diagram commutes in $\mathbf{Cat}_{\mathcal{B}_0}$

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "\epsilon"] \arrow[dr, "id_{\mathcal{B}}"'] & & \mathcal{C} \arrow[dl, "p"] \\
& \mathcal{B} &
\end{tikzcd}$$

Note that internal groups are split objects. The equivalence of categories stated in Theorem 11.4.3 from [93] can be adapted to show that there is an equivalence of categories

$$\mathbf{NatSys}_{\nu}(\mathcal{B}, \mathbf{Set}_*) \simeq \mathbf{Split}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}).$$

12.1.2. Composition pairings for natural homotopy. Now we describe the composition pairings for natural homotopy functors.

12.1.3. Theorem ([15]). *The natural system of pointed sets $\overrightarrow{P}_1(\mathcal{X})$ admits a composition pairing ν given, for all composable traces $f : x \to y, g : y \to z$ of $\mathcal{X}$, by*

$$\nu_{f,g}([f'], [g']) = [f' \star g']$$

for any $[f']$ in $\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y), f)$ and $[g']$ in $\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y, z), g)$.

Proof. Observe that the maps $\nu_x : \{*\} \to \overrightarrow{P}_1(\mathcal{X})$ for x in X are uniquely determined since the singleton is the initial object in $\mathbf{Set}_*$. For composable traces f and g of $\mathcal{X}$, the maps $\nu_{f,g}$ are well defined and are morphisms of $\mathbf{Set}_*$. Thus, we only have to check the cocycle, unit, and naturality conditions. The cocycle condition is a consequence of the fact that the composition is associative. The right unit condition is verified, since for $f : x \to y$, the following diagram

$$\begin{tikzcd}
\overrightarrow{P}_1(X)_f & & \overrightarrow{P}_1(X)_f \times \overrightarrow{P}_1(X)_{1_y} \arrow[ll, "\nu_{f,1_y}"'] \\
& & \overrightarrow{P}_1(X)_f \times \{*\} \arrow[ull, "\cong"] \arrow[u, "id_{\overrightarrow{P}_1(X)_f} \times \nu_y"']
\end{tikzcd}$$

commutes. Indeed, if c_y denotes the constant path equal to y, we have $[f] = [f \star c_y] = [f \circ \nu_y(*)]$, since $[c_y]$ is the pointed element of $\overrightarrow{P}_1(X)_{1_y}$. The left unit condition is similarly verified. Finally, the naturality condition follows from the associativity of concatenation of traces. Indeed, the equality

$$[(u \star f) \star (g \star v)] = [u \star (f \star g) \star v]$$

holds for any traces u, v, f, g of $\mathcal{X}$ such that the composites are defined. □

12.1.4. Theorem ([15]). *For every $n \geq 2$, the natural system of groups $\overrightarrow{P}_n(\mathcal{X})$ admits a composition pairing ν defined by*

$$\nu_{f,g}(\sigma, \tau) = \sigma \star \tau,$$

for all composable traces $f : x \to y$ *and* $g : y \to z$ *of* $\mathcal{X}$ *and homotopy classes* σ *in* $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ *and* τ *in* $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z), g)$*, where* $\sigma \star \tau$ *denotes the homotopy class in* $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z)$ *of the map* $t \mapsto \sigma(t) \star \tau(t)$*.*

Proof. First observe that the maps ν_x, for x in X, are uniquely determined since the trivial group is the initial object in **Gp**. Let us prove that $\overrightarrow{P}_n(\mathcal{X})$ verifies the commutator condition recalled in Proposition 11.3.5 . Given composable 1-cells f and g of $\overrightarrow{\mathbf{P}}(\mathcal{X})$, the 1-cell $(1,g)$ of $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ induces a map

$$\overrightarrow{P}_n(\mathcal{X})(1,g) : \pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f) \to \pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z), f \star g)$$

sending a class σ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ to the homotopy class of the map $t \mapsto \sigma(t) \star g$, denoted by $\sigma \star g$. We obtain a similar homomorphism from the 1-cell $(f,1)$, sending τ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z), g)$ to the homotopy class of the map $t \mapsto f \star \tau(t)$, denoted by $f \star \tau$.

Let σ, σ' in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ and τ, τ' in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z), g)$. The following exchange relation

$$(\sigma \star \tau) \cdot (\sigma' \star \tau') = (\sigma \cdot \sigma') \star (\tau \cdot \tau'),$$

where $\cdot$ denotes the product of homotopy classes in homotopy groups, holds in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z), f \star g)$. Using this relation, we have

$$(\sigma \star g) \cdot (f \star \tau) = (\sigma \cdot f) \star (g \cdot \tau) = \sigma \star \tau = (f \cdot \sigma) \star (\tau \cdot g) = (f \star \tau) \cdot (\sigma \star g).$$

for all σ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ and τ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z), g)$. We conclude via the commutator condition that $\overrightarrow{P}_n(\mathcal{X})$ admits a composition pairing, given by $\nu_{f,g}(\sigma, \tau) = \sigma \star \tau$. □

12.1.5. Theorem ([15]). *Let* $\mathcal{X} = (X, dX)$ *be a dispace. For each* $n \leq 1$ *(resp.* $n \geq 2$*) there exists a split object* $\mathcal{C}^n_{\mathcal{X}}$ *(resp. internal group* $\mathcal{C}^n_{\mathcal{X}}$*) in* $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ *such that*

$$\overrightarrow{P}_n(\mathcal{X})_f = (\mathcal{C}^n_{\mathcal{X}})_f,$$

for all traces f *of* $\mathcal{X}$*, and this assignment is functorial in* $\mathcal{X}$*.*

Proof. Using the equivalences of categories recalled in 12.1.1 (resp. in Theorem 11.4.3), and Theorem 12.1.3 (resp. Theorem 12.1.5) we obtain a split object $\mathcal{C}^1_{\mathcal{X}}$ (resp. an internal group $\mathcal{C}^n_{\mathcal{X}}$) in $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ associated to $\overrightarrow{P}_1(\mathcal{X})$ (resp. $\overrightarrow{P}_n(\mathcal{X})$ for $n \geq 2$). Let us prove that this assignment defines a functor

$$\mathcal{C}^n_{-} : \mathbf{dTop} \to \mathbf{Cat}.$$

Any morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of dispaces induces continuous maps $\varphi_{x,y} : \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{Y})(\varphi(x), \varphi(y))$ for all points x, y in X such that $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \neq \emptyset$. We define a functor $\mathcal{C}^n_{\varphi} : \mathcal{C}^n_{\mathcal{X}} \to \mathcal{C}^n_{\mathcal{Y}}$ on a 0-cell x and a 1-cell (σ, f) of $\mathcal{C}^n_{\mathcal{X}}$ by setting $\mathcal{C}^n_{\varphi}(x) = \varphi(x)$, and

$$\mathcal{C}^n_{\varphi}(\sigma, f) = (\pi_{n-1}(\varphi_{x,y})(\sigma), \overrightarrow{\mathbf{P}}(\varphi)(f)).$$

Functoriality follows from that of π_{n-1} and $\overrightarrow{\mathbf{P}}$. □

12.1.6. Natural homotopy as a split object or internal group. Let us describe the categories $\mathcal{C}^n_{\mathcal{X}}$ for $n \geq 0$. The 0-cells of $\mathcal{C}^n_{\mathcal{X}}$ are the points of X, and the set of 1-cells of $\mathcal{C}^n_{\mathcal{X}}$ with source x and target y is given by

$$\mathcal{C}^n_{\mathcal{X}}(x,y) = \coprod_{f \in \overrightarrow{\mathbf{P}}(\mathcal{X})(x,y)} \overrightarrow{P}_n(\mathcal{X})_f.$$

The projection p onto the second factor extends the category $\mathcal{C}^n_{\mathcal{X}}$ into an object of $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$.

For $n \leq 1$, the functor p is split by $\epsilon_n : \overrightarrow{\mathbf{P}}(X) \to \mathcal{C}^n_{\mathcal{X}}$ defined on any trace f on $\mathcal{X}$ by $\epsilon_n(f) = ([f], f)$. Note that for any trace f, $\overrightarrow{P}_0(\mathcal{X})_f = \{\, [f] \,\}$, hence $\epsilon_0(\overrightarrow{\mathbf{P}}(X)) = \mathcal{C}^0_{\mathcal{X}}$. The composition is defined by

$$([f'], f)([g'], g) = ([f' \star g'], f \star g),$$

for all $[f'] \in \overrightarrow{P}_n(\mathcal{X})_f$ and $[g'] \in \overrightarrow{P}_n(\mathcal{X})_g$. Note that $\mathcal{C}^0_{\mathcal{X}}$ is isomorphic to $\overrightarrow{\mathbf{P}}(\mathcal{X})$.

For $n \geq 2$, the functor p is split by the identity map $\eta : \overrightarrow{\mathbf{P}}(X) \to \mathcal{C}^n_{\mathcal{X}}$ defined by $\eta(f) = (1_{D_f}, f)$, where 1_{D_f} is the homotopy class of the constant loop equal to f. The inverse map is given by the inverse in each homotopy group, that is $(\sigma, f)^{-1} = (\sigma^{-1}, f)$. Recall that the product in $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ is the fibred product over $\overrightarrow{\mathbf{P}}(\mathcal{X})$, so we can use the internal multiplication in each homotopy group to define the multiplication map μ by setting $\mu((\sigma, f), (\sigma', f)) = (\sigma \cdot \sigma', f)$. The composition of (σ, f) and (τ, g), for homotopy classes σ and τ above f and g respectively, is given by

$$(\sigma, f) \star_0 (\tau, g) = (\nu_{f,g}(\sigma, \tau), f \star g) = (\sigma \star \tau, f \star g).$$

12.1.7. Natural homology as internal group. Recall from Remark **??** that as a consequence of the commutation condition and the triviality of the compatibility criterion for natural transformations, the categories $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathbf{Ab})$ and $\mathbf{NatSys}_\nu(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathbf{Ab})$ coincide. For all $n \geq 1$, the natural system $\overrightarrow{H}_n(\mathcal{X})$ is thus equipped with a composition pairing, and via the equivalence

$$\mathbf{Ab}(\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})) \cong \mathbf{NatSys}_\nu(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathbf{Ab})$$

we obtain an internal abelian group $\mathcal{A}^n_{\mathcal{X}}$ in the category $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. Moreover, using similar arguments as in the proof of Theorem 12.1.5, one proves that the assignment $\mathcal{A}^n_{-} : \mathbf{dTop} \to \mathbf{Cat}$ is functorial for all $n \geq 1$.

12.1.8. Fundamental category of a dispace. The *fundamental category* of a dispace $\mathcal{X}$, denoted by $\overrightarrow{\mathbf{\Pi}}(\mathcal{X})$, is the homotopy category of $\overrightarrow{\mathbf{P}}(\mathcal{X})$ when interpreted as a 2-category. Explicitly, the trace category $\overrightarrow{\mathbf{P}}(\mathcal{X})$ can be extended into a $(2,1)$-category by adding 2-cells corresponding to dihomotopies of traces. The fundamental category is the quotient of this $(2,1)$-category by the congruence generated by these 2-cells. We refer the reader to [56, 61] for a fuller treatment of fundamental categories of dispaces. This assignment defines a functor

$$\overrightarrow{\mathbf{\Pi}} : \mathbf{dTop} \to \mathbf{Cat}.$$

Given a dispace $\mathcal{X}$, consider the quotient functor $\pi : \overrightarrow{\mathbf{P}}(\mathcal{X}) \to \overrightarrow{\mathbf{\Pi}}(\mathcal{X})$, which is the identity on 0-cells and which associates a trace f to its class $[f]$ modulo path-connectedness. Similarly to [52, Theorem 1], we have the following result.

12.1.9. Proposition ([53]). *Given a dispace $\mathcal{X}$, suppose that there exists a functorial section s of the functor $\pi : \overrightarrow{\mathbf{P}}(\mathcal{X}) \to \overrightarrow{\mathbf{\Pi}}(\mathcal{X})$. Then the natural system $\overrightarrow{P}_n(\mathcal{X})$ is trivial for all $n \geq 2$.*

Proof. We show that each trace space is contractible. Let $\overrightarrow{t}(\mathcal{X})$ (resp. $\overrightarrow{t}(\mathcal{X}) \times [0,1]$) denote the natural system of topological spaces on $\overrightarrow{\mathbf{\Pi}}(\mathcal{X})$ which associates the space $[f] \subseteq \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$ (resp. $[f] \times [0,1]$) to each class $[f] : x \to y$. For a dipath g in $[f]$, denote by $g_{|[s,r]}$ the restriction of g to the interval $[s,r] \subseteq [0,1]$. Now we define a natural transformation $H : \overrightarrow{t}(\mathcal{X}) \times [0,1] \Rightarrow \overrightarrow{t}(\mathcal{X})$ such that the component $H_{[f]}$ sends a pair $(g,s) \in [f] \times [0,1]$ to the dipath

$$H_{[f]}(g,s)(t) = \begin{cases} g(t) & t \in [0, \frac{s}{2}], \\ s([g_{|[\frac{s}{2}, 1-\frac{s}{2}]}]) & t \in [\frac{s}{2}, 1 - \frac{s}{2}], \\ g(t) & t \in [1 - \frac{t}{2}, 1]. \end{cases}$$

Then $H_{[f]}(g,-)$ is a homotopy from g to $s([f])$ for every g in $[f]$. Thus every connected component of every trace space of $\mathcal{X}$ is contractible. □

12.1.10. Remarks. Recall that the homotopy groups $\pi_n(X,x)$ and $\pi_n(X,y)$ of a topological space X are isomorphic for any path-connected points x and y of X. In the definition of natural homotopy we consider the homotopy groups of trace spaces $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$ based at each trace f. However, choosing a single base-point in each connected component of each trace space of a dispace $\mathcal{X}$ requires a section as described above. Furthermore, for such a section to give rise to a natural system, it must be functorial. In this case, the only non-trivial homotopy functor is $\overrightarrow{P}_1(\mathcal{X})$, and this homotopic information is provided by $\overrightarrow{\mathbf{\Pi}}(\mathcal{X})$: the hom-set $\overrightarrow{\mathbf{\Pi}}(\mathcal{X})(x,y)$ is equal to $\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y))$.

Finally, note that natural homology decomposes, for any trace $f : x \to y$ on $\mathcal{X}$, into

$$\overrightarrow{H}_n(\mathcal{X})_f \cong \bigoplus_{[f] \in \overrightarrow{\mathbf{\Pi}}(\mathcal{X})(x,y)} H_{n-1}([f])$$

where $H_{n-1}([f])$ is the $(n-1)^{th}$ singular homology of the connected space $[f] \subset \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$.

12.2. TIME-REVERSAL INVARIANCE

In this section we study the effect of reversal of time on homotopical and homological invariants of dispaces. First, in Subsection 12.2.1 we define the notion of time-reversed dispace and show that natural homotopy and homology are time-symmetric. We then prove the main result of this section, Theorem 12.2.6 in Section 12.2.4, which states that the functors $\mathcal{C}^n_-$ and $\mathcal{A}^n_-$ are time-reversal.

12.2.1. Time-reversal in dispaces. Given a dispace $\mathcal{X} = (X, dX)$, for any dipath f in dX, we denote by $f^\sharp$ the dipath defined by

$$f^\sharp(t) = f(1-t),$$

for all t in $[0,1]$. We define its *time-reversed dispace*, or *opposite dispace*, as the dispace $\mathcal{X}^\sharp = (X, dX^\sharp)$ where $dX^\sharp$ is defined by

$$dX^\sharp = \{f^\sharp \mid f \in dX\}.$$

Note that $dX^\sharp$ is easily verified to be a set of directed paths according to the conditions listed in 11.6.1. This defines a functor $(-)^\sharp : \mathbf{dTop} \to \mathbf{dTop}$, sending a dispace $\mathcal{X}$ to its opposite. Notice that if $\phi : \mathcal{X} \to \mathcal{Y}$ is a morphism of dispaces, this functor leaves the continuous map $\phi : X \to Y$ unchanged, since $(\phi_* f)^\sharp = \phi_*(f^\sharp)$.

12.2.2. Time-reversal properties. A dispace $\mathcal{X}$ is called *time-symmetric* if the dispaces $\mathcal{X}$ and $\mathcal{X}^\sharp$ are isomorphic. In that case, by functoriality of $\overrightarrow{\mathbf{P}}$ and $\mathcal{C}^n_-$, there exist covariant isomorphisms

$$\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\sim} \overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp), \qquad \overrightarrow{P}_n(\mathcal{X}) \xrightarrow{\sim} \overrightarrow{P}_n(\mathcal{X}^\sharp), \qquad \text{and} \qquad \mathcal{C}^n_{\mathcal{X}} \xrightarrow{\sim} \mathcal{C}^n_{\mathcal{X}^\sharp}.$$

A dispace $\mathcal{X} = (X, dX)$ is called *time-contractible* when $dX = dX^\sharp$. In that case any dipath is reversible, that is $f \in dX$ implies $f^\sharp \in dX$. Note that for a dispace $\mathcal{X} = (X, dX)$, $dX = X^{[0,1]}$ implies that $\mathcal{X}$ is time-contractible but the converse is not true in general. Thus the directed homotopy of a time-contractible dispace $\mathcal{X}$ does not necessarily coincide with the homotopy of its underlying space X as shown in Proposition 11.6.5.

A functor $F : \mathbf{dTop} \to \mathbf{V}$ is *time-symmetric with respect to a category* $\mathbf{V}$ if the following diagram

$$\begin{tikzcd}
\mathbf{dTop} \arrow[r, "F"] \arrow[d, "(-)^\sharp"'] & \mathbf{V} \arrow[d, "||"] \\
\mathbf{dTop} \arrow[r, "F"'] & \mathbf{V}
\end{tikzcd}$$

commutes up to isomorphism. Such a functor is *strongly time-symmetric with respect to* $\mathbf{V}$ if there exists a natural isomorphism $F((-)^\sharp) \Rightarrow F$. A functor $F : \mathbf{dTop} \to \mathbf{Cat}$ is *time-reversal* if the following diagram

$$\begin{tikzcd}
\mathbf{dTop} \arrow[r, "F"] \arrow[d, "(-)^\sharp"'] & \mathbf{Cat} \arrow[d, "(-)^o"] \\
\mathbf{dTop} \arrow[r, "F"'] & \mathbf{Cat}
\end{tikzcd}$$

commutes up to isomorphism. Such a functor is *strongly time-reversal* if there exists a natural isomorphism $F((-)^\sharp) \Rightarrow F(-)^o$.

12.2.3. Time-symmetry of directed homology and homotopy. Here we show that without a composition pairing, natural homotopy and natural homology do not capture time-reversal.

For any dispace $\mathcal{X}$ the equalities

$$\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp) = \overrightarrow{\mathbf{P}}(\mathcal{X})^o \qquad \text{and} \qquad \overrightarrow{\mathbf{\Pi}}(\mathcal{X}^\sharp) = \overrightarrow{\mathbf{\Pi}}(\mathcal{X})^o$$

hold in **Cat**, hence the functors $\overrightarrow{\mathbf{P}}$ and $\overrightarrow{\mathbf{\Pi}}$ are strongly time-reversal. The functor which sends a dispace $\mathcal{X}$ to $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ is strongly time-symmetric with respect to **Cat**. Indeed, the isomorphism of categories

$$F^\sharp : F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to F(\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$$

sending a trace f to its opposite $f^\sharp$ and a 1-cell of (u, v) in $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ to the 1-cell $(v^\sharp, u^\sharp)$, is the component at $\mathcal{X}$ of a natural isomorphism. Note that the functors $\overrightarrow{P}_n$ and $\overrightarrow{H}_n$ are not strongly time-symmetric with respect to **opNat**(**Act**).

Consider the category **Diag**(**Act**), whose objects are the pairs $(\mathcal{C}, F)$, where $\mathcal{C}$ is a category and $F : \mathcal{C} \to \mathbf{Act}$ is a functor, and whose morphisms are pairs $(\Phi, \tau) : (\mathcal{C}, F) \to (\mathcal{C}', F')$, where $\Phi : \mathcal{C} \to \mathcal{C}'$ is a functor and $\tau : F \to F'\Phi$ is a natural transformation, with natural composition. By definition, the functors $\overrightarrow{P}_n$ and $\overrightarrow{H}_n$ are strongly time-symmetric with respect to **Diag**(**Act**).

For $n \geq 0$, we compare the functors $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{P}_n(\mathcal{X}^\sharp)$ in $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathbf{Act})$ by precomposing the latter with the isomorphism $F^\sharp$. Observe that, for all points x, y in X, we have homeomorphisms

$$\alpha_{x,y} : \overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y) \to \overrightarrow{\mathfrak{T}}(\mathcal{X}^\sharp)(y, x)$$

sending a trace f to its opposite $f^\sharp$. These induce group isomorphisms $\overrightarrow{P}_n(\mathcal{X})_f \xrightarrow{\sim} \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp}$ for all $n \geq 2$. By definition, $(F^\sharp)^*\overrightarrow{P}_n(\mathcal{X}^\sharp)_f = \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp}$, so we get components of a natural isomorphism

$$\begin{gathered}\alpha_f : \overrightarrow{P}_n(\mathcal{X})_f \longrightarrow (F^\sharp)^*\overrightarrow{P}_n(\mathcal{X}^\sharp)_f \\ [\sigma] = [(s,t) \mapsto \sigma_s(t)] \longmapsto [(s,t) \mapsto \sigma_s(1-t)] =: [\sigma^\sharp],\end{gathered}$$

where s is the parameter for the loop in the trace space, and t is the parameter for the dipath σ_s. Thus the pair $(F^\sharp, \alpha)$ is an isomorphism in the category $\mathbf{Diag}(\mathbf{Gp})$. Such an isomorphism can similarly be established in the category $\mathbf{Diag}(\mathbf{Set}_*)$ for natural homotopy in the case $n = 1$. The functor $F^\sharp$ and the isomorphisms are components at $\mathcal{X}$ of natural isomorphisms, hence $\overrightarrow{P}_n$ is strongly time-symmetric with respect to $\mathbf{Diag}(\mathbf{Act})$ for all $n \geq 1$.

A corresponding isomorphism for natural homology, $\overrightarrow{H}_n(\mathcal{X}) \cong \overrightarrow{H}_n(\mathcal{X}^\sharp)$, can be similarly established in $\mathbf{Diag}(\mathbf{Ab})$ using the functor $F^\sharp$ and the homeomorphisms $\alpha_{x,y}$, showing that $\overrightarrow{H}_n$ is strongly time-symmetric with respect to $\mathbf{Diag}(\mathbf{Ab})$ for all $n \geq 1$.

12.2.4. Time-reversibility of natural homotopy. Here we show that equipping natural homotopy and natural homology with composition pairings has solved the problem of non-detection of time-reversal. This is expressed by Theorems 12.2.5 and Theorem:MainTheoremA.

Following Theorem 12.1.5, the category $\mathcal{C}^n_{\mathcal{X}}$ with the projection $p : \mathcal{C}^n_{\mathcal{X}} \to \overrightarrow{\mathbf{P}}(\mathcal{X})$ onto the second factor is an internal group in $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. On the other hand, the category $\mathcal{C}^n_{\mathcal{X}^\sharp}$ obtained from the natural system $\overrightarrow{P}_n(\mathcal{X}^\sharp)$ via the construction given in 12.1 has 0-cells $x \in X$, while 1-cells are of the form $(\sigma^\sharp, f^\sharp) : y \to x$ where $\sigma^\sharp \in \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp}$ and $f^\sharp : y \to x$ is a trace in $\mathcal{X}^\sharp$. Composition is given by

$$(\tau^\sharp, g^\sharp) \star_0^{\mathcal{C}^n_{\mathcal{X}^\sharp}} (\sigma^\sharp, f^\sharp) = (\tau^\sharp \star \sigma^\sharp, g^\sharp \star f^\sharp).$$

We denote the associated projection by $p^\sharp$. We define for $n \geq 2$

$$I_n(\mathcal{X}) : \mathcal{C}^n_{\mathcal{X}^\sharp} \to (\mathcal{C}^n_{\mathcal{X}})^o,$$

the isomorphism of categories which is the identity on 0-cells, and which sends a 1-cell $(\sigma^\sharp, f^\sharp)$ of $\mathcal{C}^n_{\mathcal{X}^\sharp}$ to $(\sigma, f)^o$. The functoriality of $I_n(\mathcal{X})$ follows from the equality

$$(\tau^\sharp, g^\sharp) \star_0^{\mathcal{C}^n_{\mathcal{X}^\sharp}} (\sigma^\sharp, f^\sharp) = (\tau^\sharp \star \sigma^\sharp, g^\sharp \star f^\sharp) = ((\sigma \star \tau)^\sharp, (f \star g)^\sharp).$$

The opposite group $(\mathcal{C}^n_{\mathcal{X}})^o$ can be interpreted as an internal group in $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$ by composing the projection $p^o : (\mathcal{C}^n_{\mathcal{X}})^o \to \overrightarrow{\mathbf{P}}(\mathcal{X})^o$ with the canonical isomorphism $\overrightarrow{\mathbf{P}}(\mathcal{X})^o \simeq \overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$. We denote by $\widetilde{p}^o$ this composition. Then the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}^n_{\mathcal{X}^\sharp} \arrow[rr, "I_n(\mathcal{X})"] \arrow[dr, "p^\sharp"'] & & (\mathcal{C}^n_{\mathcal{X}})^o \arrow[dl, "\widetilde{p}^o"] \\
& \overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp) &
\end{tikzcd}$$

We thereby deduce that $I_n(\mathcal{X})$ is a morphism of $\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$. Furthermore, it is a group isomorphism, since the fibre groups above a 1-cell $f^\sharp$ of $\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$ are isomorphic:

$$(\mathcal{C}^n_{\mathcal{X}})^o_{f^\sharp} = (\mathcal{C}^n_{\mathcal{X}})_f = \overrightarrow{P}_n(\mathcal{X})_f \cong \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp} = (\mathcal{C}^n_{\mathcal{X}^\sharp})_{f^\sharp}.$$

An isomophism $\mathcal{C}^1_{\mathcal{X}^\sharp} \cong (\mathcal{C}^1_X)^o$ can similarly be established in the category $\mathbf{Split}(\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$. We have thus proved the following result.

12.2.5. Theorem ([15]). *Given a dispace $\mathcal{X} = (X, dX)$, $\mathcal{C}^n_{\mathcal{X}^\sharp}$ and $(\mathcal{C}^n_{\mathcal{X}})^o$ are isomorphic in $\mathbf{Gp}(\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$ for all $n \geq 2$, and in $\mathbf{Split}(\mathbf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$ for $n = 1$. In particular, the functors $\mathcal{C}^n_-$ are time-symmetric for all $n \geq 1$.*

For any $n \geq 0$, the functors $I_n(\mathcal{X})$ give components of a natural transformation. Indeed, by precomposing (resp. composing) the functor $\mathcal{C}^n_-$ with $(\cdot)^\sharp$ (resp. $(\cdot)^o$), any morphism $\phi : \mathcal{X} \to \mathcal{Y}$ of dispaces yields a commuting diagram

$$\begin{tikzcd}
\mathcal{C}^n_{\mathcal{X}^\sharp} \arrow[r, "I_n(\mathcal{X})"] \arrow[d, "\mathcal{C}^n_{\phi^\sharp}"'] & (\mathcal{C}^n_{\mathcal{X}})^o \arrow[d, "(\mathcal{C}^n_{\phi})^o"] \\
\mathcal{C}^n_{\mathcal{Y}^\sharp} \arrow[r, "I_n(\mathcal{Y})"'] & (\mathcal{C}^n_{\mathcal{Y}})^o
\end{tikzcd}$$

in $\mathbf{Cat}$. Furthermore, as shown above, these components are all isomorphisms, that is there exists a natural isomorphism

$$I_n : \mathcal{C}^n_{(-)^\sharp} \Longrightarrow (\mathcal{C}^n_-)^o.$$

We have thus proved the following result.

12.2.6. Theorem ([15]). *For any $n \geq 0$, the functor $\mathcal{C}^n_- : \mathbf{dTop} \to \mathbf{Cat}$ is strongly time-reversal.*

A consequence of Theorem 12.2.6 is that for any dispace $\mathcal{X}$, the category $\mathcal{C}^n_{\mathcal{X}}$ is dual to the category $\mathcal{C}^n_{\mathcal{X}^\sharp}$. It can similarly be shown that the functor $\mathcal{A}^n_- : \mathbf{dTop} \to \mathbf{Cat}$ associated to natural homology is strongly time-reversal for all $n \geq 1$. In the particular case of a time-symmetric space $\mathcal{X}$, the category $\mathcal{C}^n_{\mathcal{X}}$ is self-dual, *i.e.* there exists a covariant isomorphism of categories

$$\mathcal{C}^n_{\mathcal{X}} \cong (\mathcal{C}^n_{\mathcal{X}})^o.$$

12.2.7. Time-reversibility with respect to opNat. The time-reversibility of a functor with values in $\mathbf{Cat}$ is expressed via duality of categories. However, given some category $\mathbf{V}$, we can define a notion of time-reversal with respect to $\mathbf{opNat}(\mathbf{V})$ which is compatible with the interpretation of natural systems with composition pairings as categories when $\mathbf{V} = \mathbf{Act}$. Consider the functor

$$(-)^\flat : \mathbf{opNat}(\mathbf{V}) \to \mathbf{opNat}(\mathbf{V})$$

which sends a pair $(\mathcal{C}, D)$ to the pair $(\mathcal{C}^o, (F^o)^* D)$, where $F^o : F(\mathcal{C}^o) \to F\mathcal{C}$ is the covariant functor sending a 0-cell f^o of $F\mathcal{C}^o$ to f, and a 1-cell (v^o, u^o) to (u, v). To a morphism

$$(\Phi, \alpha) : (\mathcal{C}, D) \to (\mathcal{C}', D')$$

of $\mathbf{opNat}(\mathbf{V})$, the functor $(-)^\flat$ associates the morphism (Φ^o, α^o), where Φ^o is the opposite functor $\mathcal{C}^o \to (\mathcal{C}')^o$, and where the component $\alpha^o_{f^o}$ at f^o a 1-cell of $\mathcal{C}^o$ is the component α_f of α at f.

Then for F a functor $\mathbf{dTop} \to \mathbf{opNat}(\mathbf{V})$, we say that F is *time-reversal with respect to* $\mathbf{opNat}(\mathbf{V})$ if the following diagram

$$\begin{tikzcd}
\mathbf{dTop} \arrow[r, "F"] \arrow[d, "(-)^\sharp"'] & \mathbf{opNat}(\mathbf{V}) \arrow[d, "(-)^\flat"] \\
\mathbf{dTop} \arrow[r, "F"'] & \mathbf{opNat}(\mathbf{V})
\end{tikzcd}$$

commutes up to isomorphisms of the form (id, α). Explicitly, this means that if $F(\mathcal{X}) = (\mathcal{C}, D)$, then $F(\mathcal{X}^\sharp) = (\mathcal{C}^o, D')$ with $(F^o)^* D$ naturally isomorphic to D'.

Given F a functor $\mathbf{dTop} \to \mathbf{opNat}_\nu(\mathbf{Act})$, we can extend to the following diagram

$$\begin{tikzcd}
\mathbf{dTop} \arrow[r, "F"] \arrow[d, "(-)^\sharp"'] & \mathbf{opNat}_\nu(\mathbf{Act}) \arrow[r, "\mathcal{E}"] \arrow[d, "(-)^\flat"] & \mathbf{Cat} \arrow[d, "(-)^o"] \\
\mathbf{dTop} \arrow[r, "F"'] & \mathbf{opNat}_\nu(\mathbf{Act}) \arrow[r, "\mathcal{E}"'] & \mathbf{Cat}
\end{tikzcd} \tag{12.2.1}$$

where the functor $\mathcal{E} : \mathbf{opNat}(\mathbf{Act}) \to \mathbf{Cat}$ sends a pair $(\mathcal{C}, D, \nu)$ to the category in $\mathbf{Cat}_{\mathcal{C}_0}/\mathcal{C}$ defined using the constructions described in Theorem 11.4.3 and Section 12.1.1. The rightmost square commutes strictly. Indeed, denoting by $\mathcal{E}_{(\mathcal{C},D,\nu)}$ the category obtained from the natural system (D, ν) on the category $\mathcal{C}$, we have that $\mathcal{E}_{(\mathcal{C},D,\nu)^\flat}$ is the category with the same 0-cells as $\mathcal{C}^o$ and in which 1-cells are defined via the hom-sets

$$\mathcal{E}(y, x) = \coprod_{f^o \in \mathcal{C}^o(y,x)} D_f\,,$$

since by definition, $D^\flat_{f^o} = D_f$. On the other hand, $\mathcal{E}_{(\mathcal{C},D,\nu)}$ has the same 0-cells as $\mathcal{C}$ and 1-cells are defined via the hom-sets

$$\mathcal{E}(x, y) = \coprod_{f \in \mathcal{C}(x,y)} D_f\,.$$

Thus $\mathcal{E}_{(\mathcal{C},D,\nu)^\flat}$ coincides with $\mathcal{E}^o_{(\mathcal{C},D,\nu)}$. Hence, if the leftmost square in diagram (12.2.1) commutes up to isomorphism, then the outer square commutes up to isomorphism. This proves the following result.

12.2.8. Theorem ([15]). *Any functor* $F : \mathbf{dTop} \to \mathbf{opNat}_\nu(\mathbf{Act})$ *which is time-reversal with respect to* $\mathbf{opNat}(\mathbf{Act})$ *can be extended into a time-reversal functor* $\mathcal{E} \circ F : \mathbf{dTop} \to \mathbf{Cat}$.

12.3. Relative directed homotopy and exact sequences

In this section, we introduce a notion of relative homotopy for dispaces, and establish a long exact sequence, as in the case of regular topological spaces, using the homological category structure on **Act** as introduced by Grandis in [62]. See Section 11.5 for definitions and results concerning exact sequences in the category of actions.

12.3.1. Long exact sequence of relative natural homotopy. We endow the category $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathbf{Act})$ with the structure of a homological category by letting null morphisms be those natural transformations which are null component-wise in **Act**. A sequence of natural systems of actions is then *exact* when it is point-wise exact in **Act**. As a consequence we obtain the following long exact sequence of natural homotopy systems:

12.3.2. Theorem ([15]). *Let $\mathcal{X}$ be a dispace and $\mathcal{A}$ be a directed subspace of $\mathcal{X}$. There is an exact sequence in $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathbf{Act})$:*

$$\cdots \to \overrightarrow{P}_n(\mathcal{A}) \to \overrightarrow{P}_n(\mathcal{X}) \to \overrightarrow{P}_n(\mathcal{X}, \mathcal{A}) \overset{\partial_n}{\to} \overrightarrow{P}_{n-1}(\mathcal{A}) \to \cdots$$
$$\cdots \to \overrightarrow{P}_2(\mathcal{A}) \overset{v}{\to} \overrightarrow{P}_2(\mathcal{X}) \overset{f}{\to} (\overrightarrow{P}_2(\mathcal{X}, \mathcal{A}), \overrightarrow{P}_2(\mathcal{X})) \overset{g}{\to} \overrightarrow{P}_1(\mathcal{A}) \overset{h}{\to} \overrightarrow{P}_1(\mathcal{X}) \to \overrightarrow{P}_1(\mathcal{X}, \mathcal{A}) \to 0.$$

12.3.3. Dicontractible subspaces. A dispace $\mathcal{X}$ is called *dicontractible* if all its natural homotopy functors $\overrightarrow{P}_n(\mathcal{X})$ are trivial, e.g. are constant functors into a singleton for $n = 1$ or a trivial group for $n \geq 2$. Following Theorem 12.3.2, if $\mathcal{A}$ is a dicontractible directed subspace of $\mathcal{X}$, then we have an isomorphism

$$\overrightarrow{P}_n(\mathcal{X}) \simeq \overrightarrow{P}_n(\mathcal{X}, \mathcal{A}),$$

in $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathbf{Gp})$ for all $n \geq 3$. Note that when (X, dX) is the geometric realization of a non-self-linked precubical set (a large class of precubical sets, in which e.g. the semantics of concurrent systems can be expressed, see [43] for more details), the dicontractibility condition is equivalent to asking that all path spaces are contractible, since, by Proposition 3.14 of [98], all its trace spaces have the homotopy type of a CW-complex.

12.3.4. A long exact fibration sequence in directed topology. Recall that a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of dispaces induces a natural tranformation $\overrightarrow{\varphi} : \overrightarrow{T}_*(\mathcal{X}) \Rightarrow \overrightarrow{T}_*(\mathcal{Y})$. We consider morphisms $p : \mathcal{E} \to \mathcal{B}$ of dispaces such that each component $\overrightarrow{p}_e$ is a fibration, for every e a dipath of $\mathcal{E}$. We define the associated *natural system of fibres*, denoted $\overrightarrow{T}_*(\mathcal{F})$, as the natural system of pointed topological spaces on $\overrightarrow{\mathbf{P}}(\mathcal{E})$ which sends a dipath e to

$$\overrightarrow{T}(\mathcal{F})_e = \left(\overrightarrow{p}_e^{-1}(p(e))\ ,\ e\right).$$

Now for each 1-cell e of $\overrightarrow{\mathbf{P}}(\mathcal{E})$, denote by $\overrightarrow{P}_n(\mathcal{F})_e$ (resp. $\overrightarrow{P}_n(\mathcal{E},\mathcal{F})_e$) the homotopy group (resp. relative homotopy group)

$$\pi_{n-1}\left(\overrightarrow{T}(\mathcal{F})_e\right) \qquad \text{(resp. } \pi_{n-1}\left(\overrightarrow{T}(\mathcal{E})_e, \overrightarrow{T}(\mathcal{F})_e\right)\text{)}.$$

These are natural systems on $\overrightarrow{\mathbf{P}}(\mathcal{E})$. Furthermore, for each e dipath of $\mathcal{E}$, the sequence

$$\overrightarrow{T}(\mathcal{F})_e \to \overrightarrow{T}(\mathcal{E})_e \to \overrightarrow{T}(\mathcal{B})_{p(e)}$$

of topological spaces induces a long exact sequence of homotopy groups. Extending this to lower-dimensional homotopy groups via [62, Theorem 6.4.9] yields the following result.

12.3.5. Theorem ([15]). *Let $p : \mathcal{E} \to \mathcal{B}$ be a morphism of dispaces inducing (Serre) fibrations $\overrightarrow{p}_e$ for every 1-cell e of $\overrightarrow{\mathbf{P}}(\mathcal{E})$. Then we obtain a long exact sequence in* $\mathbf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{E}), \mathbf{Act})$*:*

$$\cdots \to \overrightarrow{P}_n(\mathcal{F}) \to \overrightarrow{P}_n(\mathcal{E}) \to \overrightarrow{P}_n(\mathcal{E},\mathcal{F}) \to \overrightarrow{P}_{n-1}(\mathcal{F}) \to \cdots$$
$$\cdots \to \overrightarrow{P}_2(\mathcal{F}) \to \overrightarrow{P}_2(\mathcal{E}) \to \left(\overrightarrow{P}_2(\mathcal{E},\mathcal{F}) \ , \ \overrightarrow{P}_2(\mathcal{E})\right) \to \overrightarrow{P}_1(\mathcal{F}) \to \overrightarrow{P}_1(\mathcal{E}) \to \overrightarrow{P}_1(\mathcal{E},\mathcal{F}) \to 0.$$

Furthermore, $\overrightarrow{P}_n(\mathcal{E},\mathcal{F}) \cong p^*(\overrightarrow{P}_n(\mathcal{B}))$ for all $n \geq 2$. In particular, when $\overrightarrow{T}_*(\mathcal{B})_{p(e)}$ is path connected for all dipaths e of $\mathcal{E}$, the isomorphism holds for all $n \geq 1$.

12.3.6. Example. Given a morphism $p : \mathcal{E} \to \mathcal{B}$ of dispaces, if the continuous map $p : E \to B$ is a fibration, then p induces fibrations $\overrightarrow{p}_e$ of trace spaces. Indeed, for all x, y in X given maps φ and h such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "id_X \times \{0\}"'] & \overrightarrow{\mathfrak{T}}(\mathcal{E})(x,y) \arrow[d, "\overrightarrow{p}_e"] \\
X \times [0,1] \arrow[r, "h"'] & \overrightarrow{\mathfrak{T}}(\mathcal{B})(p(x),p(y))
\end{tikzcd}$$

we obtain the following commutative diagram

$$\begin{tikzcd}
X \times [0,1] \arrow[r, "\Phi"] \arrow[d, "id_{X\times[0,1]} \times \{0\}"'] & E \arrow[d, "p"] \\
(X \times [0,1]) \times [0,1] \arrow[r, "H"'] & B
\end{tikzcd}$$

where $\Phi(z,t) = \varphi(z)(t)$ and $H((z,t),s) = h_s(z)(t)$ for all z in X and s, t in $[0,1]$. Since by hypothesis the map p is a fibration, there exists a unique map $\widetilde{H} : (X \times [0,1]) \times [0,1] \to E$ such that $\widetilde{H}((z,t),0) = \Phi(z,t)$. Defining the map $\widetilde{h} : X \times [0,1] \to \overrightarrow{\mathfrak{T}}(\mathcal{E})(x,y)$ by sending (z,s) to the dipath $t \mapsto \widetilde{H}((z,t),s)$ yields a continuous map which is the unique lift of h such that $\widetilde{h}(z,0) = \varphi(z)$.

From this we deduce that given a dispace $\mathcal{B}$ and a fibration $p : E \to B$, we obtain a morphism of dispaces from $\mathcal{E}$ to $\mathcal{B}$ inducing fibrations on trace spaces by setting $\mathcal{E} = (E, dE)$ where $dE = \{e \in E^{[0,1]} \mid p \circ e \in dB\}$.

CHAPTER 13.
PERSISTENCE AND NATURAL HOMOLOGY

Geometry and Algebraic Topology have now been in the computer science landscape for many years. In topological data analysis for instance, the shape of a point-cloud can be hinted at through suitable homological invariants, known as persistent homology (see e.g. [34], and [20] for a survey of the earlier days of persistence). These invariants capture the essential features of the point-cloud data, in that these are independent to the metrics used, robust to noise, and compact in their presentation. Similar ideas appeared in the realm of semantics of programming languages, and in particular in concurrency theory [54, 55] and distributed computing [72] (see e.g. [43, 62, 73] for surveys), at about the same time or slightly before.

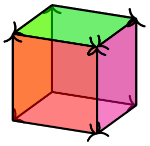
In this chapter we make a formal bridge between these two approaches and show the interest of applying persistence to problems in concurrency theory and distributed computing, though a motivational example. It contains original contributions from ongoing work. We study links between natural homology and persistent homology, the latter being tractable in the uni-dimensional case. First, in Section 13.1, we consider an example and discuss obtaining filtrations, the starting point for persistent homology, from directed spaces.

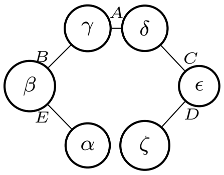
In Section 13.2, we show that natural homology is in fact a persistence object. This is essentially due to the observation that the factorisation category of the trace category of a partially ordered space is a poset; this is given in Lemma 13.2.2. Next, in Proposition 13.2.6 we show that each trace yields a uni-dimensional persistent homology module. We apply this to the motivational example in Section 13.2.7 before turning to the question of amalgamating this uni-dimensional information. This is addressed in Section 13.3, but first we prove some results concerning colimits of posets, the first of which, Proposition 13.3.2 is folklore, and the second of which, Proposition 13.3.3 refines the construction in the case of chains.

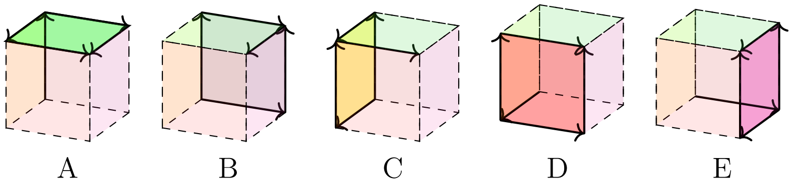
Next, in Section 13.3.4, we apply similar colimit constructions to functors whose domains are posets and whose codomains are a fixed category $\mathcal{C}$. This gives Proposition 13.3.5, which states that a functor $D : P \to \mathcal{C}$ from a poset P to a cocomplete category $\mathcal{C}$ may be calculated as the colimit of its restrictions to subposets when the colimit of the subposets gives P. As corollaries of the preceding propositions, we obtain the main theorem of this chapter, which states that the natural homology of partially ordered spaces is recovered as a colimit of persistence modules in various ways. This is given by Theorems 13.3.7.

13.1. THE MATCHBOX EXAMPLE

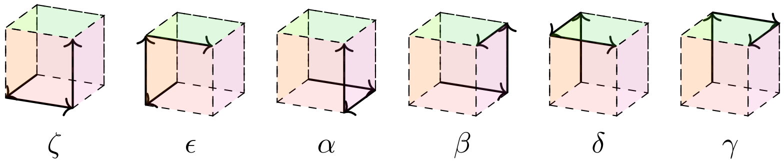
To give a glimpse of the intimate relationship between multidimensional persistence and natural homology, let us describe our construction on Fahrenberg's matchbox example [40], pictured left below (all but the bottom face of the unit cube $[0,1] \times [0,1] \times [0,1]$ is in the cubical complex, *i.e.* there are 5 squares glued together). Recall from Section 11.6.1 that a trace is the equivalence class $\overline{p}$ of a dipath p modulo monotonic and continuous reparametrization, and the set of such equivalence classes can be given the structure of a topological space $\overrightarrow{\mathfrak{T}}(K)(a,b)$ for all start (resp. end) points a (resp. b), homotopic to a CW-complex for a large class of geometric realizations of pre-cubical sets. Using Ziemiański's construction [119], the CW-complex (or simplicial set as well in this case) corresponding to its trace space from beginning to end is shown below, where the edges A, B, C, D and E correspond to the 5 2-dimensional cubical paths shown in the picture below :







and the vertices α, β, γ, δ, ϵ and ζ correspond to the 6 1-dimensional dipaths :

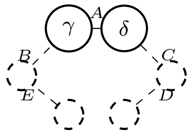
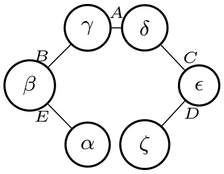
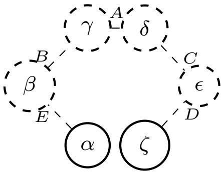


Note that β is the geometric intersection of B with E, γ is the geometric intersection of B with A etc. leading indeed to the simplicial set pictured right-hand side of the first figure.

In order to apply persistent homology, we need to obtain a (multi-dimensional) filtration from this directed space, *i.e.* a map from some poset P such as $\mathbb{N}^2$ to the category of simplicial sets. One could think that such a filtration could be obtained by moving the end-points a and b associated to a trace space $\overrightarrow{\mathfrak{T}}(K)(a,b)$. However, as we illustrate below, there is no canonical way of obtaining such a filtration in general; we must use extensions along traces to define inclusion maps.

There are maps from $\overrightarrow{\mathfrak{T}}(K)(a,b)$ to $\overrightarrow{\mathfrak{T}}(K)(a,')b'$, for $a \leq a'$ and $b' \leq b$ that act as restriction maps : they just "cut" the combinatorial dipaths so as to only keep the parts (if any) that go from a' to b'. Hence, we get a decreasing sequence of simplicial sets as soon as any of the three coordinates of a increase or any of the three coordinates of b decrease. Below, we have represented the part of the multidimensional filtration generated, for the vertical coordinate of b (the end point) and of a (the starting point) ; recall also that the 5 squares are here unit squares and the lower coordinates are 0, upper ones are 1. In this filtration, the restriction maps acting on combinatorial dipaths should correspond to inclusion maps from bottom to top, and from left to right, of simplicial sets representing the corresponding trace spaces.

For instance, moving the end point b from vertical coordinate 1 to 0 while keeping vertical coordinate of a at 0 (right column in the table below), the only 1-dimensional paths going through coordinate 0 for b are α and ζ, hence all other vertices (and edges) have to disappear. This induces the upwards inclusion map from the two point simplicial set (α and ζ) into the connected simplicial set above : H_0 of these simplicial sets goes from $\mathbb{Z}^2$ to $\mathbb{Z}$, "killing" one component when extending paths to reach the end point of the matchbox. This corresponds, in the natural homology diagram $\overrightarrow{H}_1(K)$, to part of the diagram being a projection map from $\mathbb{Z}^2$ to $\mathbb{Z}$ when moving b to the endpoint of the matchbox, while keeping the starting point fixed at the initial vertex.

b/a	1	0
1		
0	∅	

The reason that we obtain an inclusion map from bottom to top in this case is because there is a unique map from the point $(1,1,0)$ to the point $(1,1,1)$. When there is a choice between maps, we no longer obtain a canonical inclusion map. Indeed, consider the case in which $a = (0,0,0)$, the initial point, $b' = (1,1,0)$ and $b = (1,1,1)$, the terminal point. There are two extension maps from $\overrightarrow{\mathfrak{T}}(K)(b',b)$ to $\overrightarrow{\mathfrak{T}}(K)(a,b)$, one by precomposition by the trace of ζ, and one by precomposition by the trace of α. These will produce different homology maps once the invariant is applied.

Therefore there is no canonical multi-dimensional filtration of the trace spaces which depends only on start and end points. More generally, it is easily seen that such a multi-dimensional filtration for studying a directed space X would exist only if we had a way to associate in a continuous manner, to each pair of points α and β, a directed path going from α to β. It is well-known that indeed, such a continuous map will only exist if X is contractible in a directed manner, i.e. is trivial, see e.g. [53], and see Proposition 12.1.9.

The objective of this chapter is to cope with this difficulty, in order to give a meaning to natural homology as persistent homology. We will use the directed structure of a pospace to obtain unidimensional persistence homologies, and glue this information together to obtain the whole natural homology diagram. In many ways, this resembles "probing" approaches in multi-dimensional persistent homology, see e.g. [26].

13.2. NATURAL HOMOLOGY AS A PERSISTENCE OBJECT

13.2.1. Posets of directed paths. Consider a directed space $\mathcal{X} = (X, dX)$ and its trace category $\overrightarrow{\mathbf{P}}(\mathcal{X})$. We define a relation on traces in $\mathcal{X}$ by

$$f \leq g \qquad \Longleftrightarrow \qquad \exists\, u, v \in dX, \quad g = ufv.$$

We recall that a *pospace* $\mathcal{X} = (X, \leq_X)$ consists of a Hausdorff topological space X and a partial order $\leq_X$ which is closed in the product topology $X \times X$. Pospaces are naturally interpreted as directed spaces by equipping them with the set of increasing paths dX from the unit interval, with its usual ordering, to X.

13.2.2. Lemma. *In any directed space, this defines a pre-order. In a pospace, it is a partial order relation.*

Proof. Since constant paths are directed, the relation is reflexive, and we have transitivity by associativity of concatenation of traces. Indeed, if $f \leq g$ and $g \leq h$, there exist extensions (u, v) and (u', v') such that

$$f = ugv \qquad \text{and} \qquad g = u'hv'.$$

Thus, $f = u(u'hv')v = (uu')h(v'v)$, *i.e.* $f \leq h$. In the case of a loop-free directed space, we also need to prove anti-symmetry of $\leq$. Consider $f, g \in dX$ such that $f \leq g$ and $g \leq f$. By definition there exist extensions such that $f = ugv$ and $g = u'fv'$. Thus $f = uu'fv'v$, so uu' and vv' are loops, and must therefore be constant paths. By Theorem 13.2.3 below, this means that the image of both uu' and $v'v$ are the singleton space, meaning that the image of each of the dipaths u, u', v, v' are the singleton space, *i.e.* these are all constant paths, concluding the proof. ☐

This poset, denoted by $\mathbb{P}(X)$, will be called the *trace poset* of $(X, \leq_X)$. Consider a pospace $\mathcal{X} = (X, \leq)$. We know from [71] that dipaths in $\mathcal{X}$ are characterized by their image:

13.2.3. Theorem (Thm. 3.15 of [71]). *The image of a dipath in a pospace is isomorphic to either the directed unit interval or the singleton space.*

The above results essentially state that two dipaths are equal modulo reparametrisation if and only if they have the same image, and that this induces a partial ordering on traces. In particular, we obtain the following result:

13.2.4. Proposition. *For a pospace $\mathcal{X}$, $\mathbb{P}(\mathcal{X})$ is isomorphic to $F\overrightarrow{\mathbf{P}}(\mathcal{X})$.*

Proof. By the above theorem, we have

$$ufv = u'fv' \qquad \Longleftrightarrow \qquad u = u' \text{ and } v = v',$$

meaning that there is at most one extension between any two traces. □

This allows us to interpret natural homology as a functor on a poset, *i.e.* as a persistence object. Indeed, in a pospace $\mathcal{X}$, the i^{th} natural homology diagram associated to $\mathcal{X}$ is a $\mathbb{P}(\mathcal{X})$-persistence group. Taking coefficients in a field $\mathbb{K}$, we obtain $\mathbb{P}(\mathcal{X})$-persistence $\mathbb{K}$-vector spaces.

13.2.5. Persistent homology along a trace. Let $\overline{f}$ be a trace in $\mathcal{X}$ a pospace for which we have fixed some parametrisation f. For a given point α_f in the image of f, denote by $[\alpha_f, f]$ the interval between α_f, identified with the constant trace $f(t_0) = \alpha_f$ for some $t_0 \in [0,1]$, and f in the poset $\mathbb{P}(\mathcal{X})$. In other words, $[\alpha_f, f]$ corresponds to traces p in $\mathbb{P}(\mathcal{X})$, such that $\alpha_f \leq p \leq f$.

Given parametrisations of $[0,t_0]$ and $[t_0,1]$ within $[0,1]$ via maps $\gamma_- : \ [0,1] \to [0,t_0]$ and $\gamma_+ : \ [0,1] \to [t_0,1]$ respectively, we denote by ${}_sf_s$ the trace of f restricted to $[t_0 - \gamma_-(s), t_0 + \gamma_+(s)]$. We thus obtain a $[0,1]$-persistence simplicial complex $\{K({}_sf_s)\}_s$ by applying the trace diagram functor. This is a filtration of the trace space associated to $\overline{f}$. Notice that we are always considering a chain in the poset of traces from the constant trace $f(0)$, $f(1)$, or $f(t_0)$ to f. We call such filtrations initial (resp. terminal) point filtrations when $\alpha_f = f(0)$ (resp. $\alpha_f = f(1)$)

Taking some order preserving map $\mathbb{N} \to [0,1]$, we obtain $\mathbb{N}$-persistence simplicial complexes from the above constructions. In all cases, we obtain a chain $c = (f_i)_{i \in \mathbb{N}}$ in the interval $[\alpha_f, f]$ and define the *persistent homology along $\overline{f}$ with respect to c* as a functor from $\mathbb{N}$, seen as the poset category (with the usual ordering) to the category of abelian groups or $\mathbb{K}$-vector spaces, associating $i \in \mathbb{N}$ to:

$$\overrightarrow{H}_k(f,c)_i := \overrightarrow{H}_k(\mathcal{X})_{f_i}$$

where we recall that $\overrightarrow{H}_k(\mathcal{X})_{f_i}$ is the natural homology in dimension k of $\mathcal{X}$ of the trace space associated to f_i.

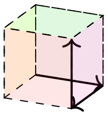
To make the construction above more clear, we will show explicitly how to find the homology along a trace by restriction of the natural homology diagram. Recall that in a pospace $\mathcal{X}$, the trace poset $\mathbb{P}(\mathcal{X})$ and the factorisation category of the trace category

$F\overrightarrow{\mathbf{P}}(X)$ are isomorphic, so we may view natural homology $\overrightarrow{H}_n(\mathcal{X})$ as a functor on $\mathbb{P}(\mathcal{X})$ with values in the category **Ab** or $\mathbf{Vect}_{\mathbb{K}}$. In practice, persistent homology is defined by taking homology with coefficients in a field $\mathbb{K}$, whereas natural homology generally considers more general abelian group coefficients. We will restrict in all practical cases to homology with coefficients in $\mathbb{K}$ so as to make comparisons possible.

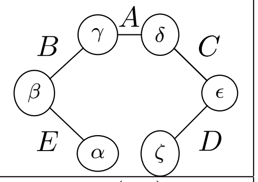
13.2.6. Proposition. *Let $\mathcal{X} = (X, dX)$ be a loop-free directed space and f a trace in $\mathcal{X}$. Let $(f_i)_{i\in\mathbb{N}}$ be a chain in the interval $[\alpha_f, f]$ in the poset of traces. Restricting the natural homology functor $\overrightarrow{H}_n(\mathcal{X})$ to the chain $(f_i)_{i\in\mathbb{N}}$, we obtain a persistence $\mathbb{K}$-vector space $\overrightarrow{H}_n(\mathcal{X})_{f_i}$ indexed by the chain $(f_i)_{i\in\mathbb{N}}$.*

13.2.7. Example: persistent homology of the matchbox. Consider the case of the matchbox example presented in Section 13.1. Let us calculate the persistent homology along each of its maximal traces, starting at the initial point and extending into the future. We fix some field $\mathbb{K}$ and will proceed via the method developed in [120]. To each of the maximal traces $\alpha, \beta, \gamma, \ldots,$ pictured in Section 13.1, we associate a sequence of subtraces corresponding to a decomposition of the trace along each 1-cube in the matchbox. For example, the trace α is decomposed into the chain

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 = \alpha,$$



where α_0 is the constant path equal to $(0,0,0)$, which α_1 extends to $(0,1,0)$. This trace is further extended to $(0,1,1)$ obtaining α_2, and then finally extended to $(1,1,1)$ giving the total trace α. A similar decomposition, which we will denote with the same indices, can be found for each of the maximal traces. This sequence of traces gives a filtration of the simplicial complex pictured above. In the case of α, we obtain the following filtration:

α_0	α_1	α_2 ζ_2	
$K(\alpha_0)$	$K(\alpha_1)$	$K(\alpha_2)$	$K(\alpha_3)$

We extend this filtration into a $\mathbb{N}$-persistence simplicial complex by considering copies of $K(\alpha_3)$ for all $i \geq 4$. We denote by $C_*(K(\alpha_i))$ the chain complex over $\mathbb{K}$ obtained from the simplicial complex $K(\alpha_i)$, obtaining a $\mathbb{N}$-persistence chain complex

$$C_*(K(\alpha_0)) \xrightarrow{f_{0,1}} C_*(K(\alpha_1)) \xrightarrow{f_{1,2}} C_*(K(\alpha_2)) \xrightarrow{f_{2,3}} C_*(K(\alpha_3)) \xrightarrow{f_{3,4}} \cdots$$

where $f_{n,n+1}$ are induced by the inclusions of simplicial sets given by the filtration. For each natural number p, we denote by H_p^i the p^{th} homology group of $C_*(K(\alpha_i))$, thus obtaining a sequence of homology groups

$$H_p^0 \xrightarrow{\phi_{0,1}} H_p^1 \xrightarrow{\phi_{1,2}} H_p^2 \xrightarrow{\phi_{2,3}} H_p^3 \xrightarrow{\phi_{3,4}} H_p^3 \xrightarrow{\phi_{4,5}} \cdots \xrightarrow{\phi_{n-1,n}} H_p^3 \xrightarrow{\phi_{n,n+1}} \cdots ,$$

where the $\phi_{n,n+1}$ are identities for $n \geq 3$. This is in fact an $\mathbb{N}$-persistence $\mathbb{K}$-vector space. We define a non-negatively graded module over $\mathbb{K}[t]$ by setting

$$H_p := \left(\bigoplus_{i=0}^{2} H_p^i \right) \oplus \left(\bigoplus_{i=3}^{\infty} H_p^3 \right),$$

and defining the action of t by $t \cdot (h_i)_i = (\phi_{i,i+1}(h_i))_i$, where the h_i belongs to H_p^i, see Section 11.7.3.

We will calculate the graded module of persistent homology via matrix representations of the boundary maps ∂_k associated to the persistence chain complex $C_*(K(\alpha_i))$ as described in [120], see Section 11.7.5. We calculate $H_0(\alpha)$, the 0^{th} persistent homology along α. For this, we fix homogeneous bases for Z_0 and $\mathcal{C}_1$. Since $Z_0 = \mathcal{C}_0$, we may take the standard basis in both cases. Thus, for $\mathcal{C}_0$ we obtain the basis $\{\alpha_0, \zeta_2, \beta, \gamma, \delta, \epsilon\}$, and for $\mathcal{C}_1$ we obtain the basis $\{A, B, C, D, E\}$. We now calculate the matrix of ∂_1 with respect to these bases, taking care to order the basis of $\mathcal{C}_0$ in reverse degree order:

$$\begin{array}{c|ccccc}
 & B & A & C & D & E \\
\hline
\beta & -1 & 0 & 0 & 0 & 1 \\
\gamma & 1 & -1 & 0 & 0 & 0 \\
\delta & 0 & 1 & -1 & 0 & 0 \\
\epsilon & 0 & 0 & 1 & -1 & 0 \\
\zeta_2 & 0 & 0 & 0 & t & 0 \\
\alpha_0 & 0 & 0 & 0 & 0 & -t^3
\end{array}$$

We now calculate the column-echelon form of the above matrix, obtaining

$$\begin{array}{c|ccccc}
 & B & A & C & D & E' \\
\hline
\beta & \boxed{-1} & 0 & 0 & 0 & 0 \\
\gamma & 1 & \boxed{-1} & 0 & 0 & 0 \\
\delta & 0 & 1 & \boxed{-1} & 0 & 0 \\
\epsilon & 0 & 0 & 1 & \boxed{-1} & 0 \\
\zeta_2 & 0 & 0 & 0 & \mathrm{t} & \boxed{t} \\
\alpha_0 & 0 & 0 & 0 & 0 & -t^3
\end{array}$$

where $E' = A + B + C + D + E$. In the case of the persistent homology along α, we see that the first four rows contribute nothing to the description of $H_0(\alpha)$, and that the last two contribute $\Sigma^2\mathbb{K}[t]/t$ and $\mathbb{K}[t]$ respectively, *i.e.*

$$H_0(\alpha) \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}^2 \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus \cdots$$

We obtain a similar result for the persistent homology along ζ. Below are the standard and column-echelon forms of ∂_1 in this case:

$$\begin{array}{c|ccccc}
 & B & A & C & D & E \\ \hline
\beta & -1 & 0 & 0 & 0 & 1 \\
\gamma & 1 & -1 & 0 & 0 & 0 \\
\delta & 0 & 1 & -1 & 0 & 0 \\
\epsilon & 0 & 0 & 1 & -1 & 0 \\
\zeta_2 & 0 & 0 & 0 & 0 & -t \\
\alpha_0 & 0 & 0 & 0 & t^3 & 0
\end{array}
\qquad
\begin{array}{c|ccccc}
 & B & A & C & D & E' \\ \hline
\beta & \boxed{-1} & 0 & 0 & 0 & 0 \\
\gamma & 1 & \boxed{-1} & 0 & 0 & 0 \\
\delta & 0 & 1 & \boxed{-1} & 0 & 0 \\
\epsilon & 0 & 0 & 1 & \boxed{-1} & 0 \\
\zeta_2 & 0 & 0 & 0 & 0 & \boxed{-t} \\
\alpha_0 & 0 & 0 & 0 & t^3 & t^3
\end{array}$$

We therefore obtain the same isomorphism class for $H_0(\zeta)$ as we did in the case of α:

$$H_0(\zeta) \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}^2 \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K} \oplus \cdots$$

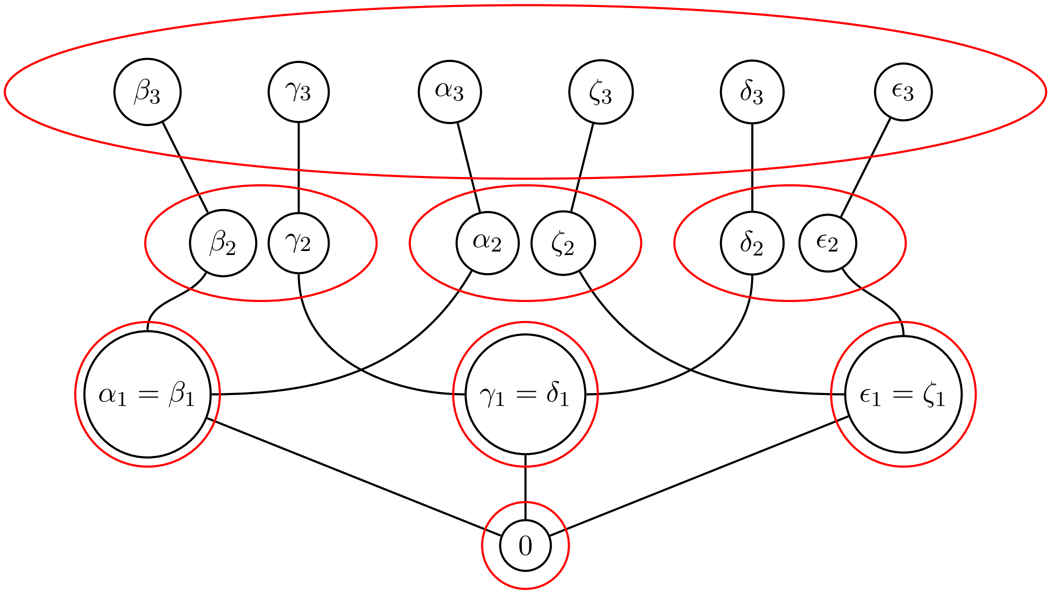
The other four dipaths yield a simple persistent homology. Indeed, there is only one class which persists throughout the sequence. Indeed, along β for example, a homogeneous basis of C_0 is $\{\beta_0, \gamma_2, \epsilon, \zeta, \delta, \alpha\}$. The basis for C_1 is the same, but now B has degree two rather than three. We obtain the following matrix representation of ∂_1:

$$\begin{array}{c|ccccc}
 & E & A & C & D & B \\ \hline
\alpha & -1 & 0 & 0 & 0 & 0 \\
\delta & 0 & 1 & -1 & 0 & 0 \\
\zeta & 0 & 0 & 0 & 1 & 0 \\
\epsilon & 0 & 0 & 1 & -1 & 0 \\
\gamma_2 & 0 & -t & 0 & 0 & 1 \\
\beta_0 & t^3 & 0 & 0 & 0 & -t^2
\end{array}
\qquad
\begin{array}{c|ccccc}
 & E & A & D & A+C & B \\ \hline
\alpha & \boxed{-1} & 0 & 0 & 0 & 0 \\
\delta & 0 & \boxed{1} & 0 & 0 & 0 \\
\zeta & 0 & 0 & \boxed{1} & 0 & 0 \\
\epsilon & 0 & 0 & -1 & \boxed{1} & 0 \\
\gamma_2 & 0 & -t & 0 & -t & \boxed{1} \\
\beta_0 & t^3 & 0 & 0 & 0 & -t^2
\end{array}$$

We indeed see that the graded module $H_0(\beta)$ is simply $\mathbb{K}[t]$, since all pivots are of degree zero and there is one non-pivot row.

13.2.8. Natural homology of the matchbox. Now we describe the restriction of the natural homology diagram of the matchbox to the principal upset in $\mathbb{P}(\mathcal{X})$ given by the constant path at the initial point $(0, 0, 0)$. The following diagram depicts the

Hasse diagram of this upset:



Traces in the same red circle yield the same trace space, *i.e.* have the same beginning and end points. Each line corresponds to an extension. The natural homology diagram is depicted below, the arrows being induced by extensions:

$$\begin{tikzcd}
 & & \mathbb{K} & & \\
\mathbb{K} \arrow[urr] & & \mathbb{K}^2 \arrow[u] & & \mathbb{K} \arrow[ull] \\
\mathbb{K} \arrow[u] \arrow[urr] & & \mathbb{K} \arrow[ull] \arrow[urr] & & \mathbb{K} \arrow[u] \arrow[ull] \\
 & & \mathbb{K} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Each path of length 3 in the above diagram corresponds to the persistence vector space obtained by taking the persistent homology along one of the maximal traces. Note that since none of the maps depicted in the diagram are identically zero, the only maps which are not uniquely defined are those with codomain $\mathbb{K}^2$.

13.3. Recovering natural homology from persistence

Here we show how the uni-dimensional persistence vector spaces we obtained along traces in the previous section may be amalgamated in order to recover the whole natural homology diagram, or subdiagrams thereof. First, we must state some facts about posets and functor categories.

13.3.1. Colimits of chains in posets. Note that in general, using exclusively maximal chains is not enough to reconstruct a poset. Indeed, consider the poset whose Hasse diagram is depicted below:

$$\begin{tikzcd}
& z & \\
y_1 \arrow[ur] & & y_2 \arrow[ul] \\
& x \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Its maximal chains are (x, y_1, z) and (x, y_2, z). In order to obtain the whole poset as a colimit, we additionally need the inclusions of x and z into these chains.

While a poset is in general, not the colimit of its maximal chains, it is the colimit of all of its chains. Given a poset P, we consider the full subcategory $\mathbf{Ch}_P$ of $\mathbf{Pos}_{in}$ consisting of the chains of P, called the *poset of chains*. The *diagram of chains* associated to P is the inclusion functor $\mathcal{F}_P : \mathbf{Ch}_P \to \mathbf{Pos}$. We have:

13.3.2. Proposition. *For any poset P,*

$$\operatorname{colim} \mathcal{F}_P = P.$$

Proof. The colimit of $\mathcal{F}_P$ is computed as follows, in $\mathbf{Pos}$: first we take the coproduct of all chains of P, which is the union of all sub-linear orders $(x_i^j)_{i \in I}$ of P where j ranges over all chains of P, for some indexing family I and $x_i^j \in P$ for all $i \in I$. Then we identify all common subchains within the $(x_i^j)_{i \in I}$, in particular the elements themselves. Therefore we identify all x_i^j within these linear orders that have to be identified, and take the transitive closure of the corresponding orders. This is indeed poset P. $\square$

When considering exclusively maximal chains, we may obtain P by adding certain intersections thereof. This can be done in two ways: either by taking full intersections or restricting to intersections which are chains. In the latter approach, the colimit is of a functor whose domain is a subcategory of $\mathbf{Ch}_P$, whereas in the former, the posets used in the colimit are not necessarily chains.

Consider a (finite) poset P and two maximal chains C_1, C_2 of P. The pullback of C_1, C_2 in $\mathbf{Pos}$ corresponds to the full sub-poset of P given by the intersection $C_1 \cap C_2$. Denote by $\mathbf{pmCh}_P$ the subcategory of $\mathbf{Pos}_{in}$ consisting of maximal chains in P and their pullbacks. By maximality of the considered chains, this category looks like a zig-zag.

A *chain quasi-pullback* of two chains C_1 and C_2 in P is a chain C_{12} such that $C_{12} \to C_1, C_2$ in $\mathbf{Pos}_{in}$ and such that for any chain $C' \to C_1, C_2$, there exists $C' \to C_{12}$ in $\mathbf{Pos}_{in}$. This is summed up in the diagram below:

$$\begin{tikzcd}
 & & C_1 \arrow[dr] & \\
C' \arrow[r, dotted, "\exists"] \arrow[urr, dashed, bend left] \arrow[drr, dashed, bend right] & C_{12} \arrow[ur] \arrow[dr] & & P \\
 & & C_2 \arrow[ur] &
\end{tikzcd}$$

The category $\mathbf{Ch}_P$ of chains in P is closed under quasi-pullback. Indeed, the poset induced by the intersection C_1 and C_2 is a collection of chains. The maximal chains of this intersection are precisely the quasi-pullbacks of C_1 and C_2. Consider the (discrete) subcategory of $\mathbf{Pos}_{in}$ consisting of the maximal chains in P. The completion of this category by chain quasi-pullbacks is denoted by $\mathbf{mCh}_P$. It is a subcategory of $\mathbf{Ch}_P$.

Compiling all of this, we obtain the following result:

13.3.3. Proposition. *A poset P is the colimit of the following inclusion functors:*

– *The minimal (completed) diagram of maximal chains*

$$\mathbf{pmCh} \to \mathbf{Pos}.$$

– *The (completed) diagram of maximal chains*

$$\mathbf{mCh} \to \mathbf{Pos}.$$

Proof. These follow the same schema as the proof of Proposition 13.3.2. □

13.3.4. Application to diagrams. Now that we have assembled information on colimits of chains, we will see how these constructions carry over to colimits of diagrams over chains. We fix a category $\mathcal{C}$.

First, we introduce a category representing persistence objects of a certain type $\mathcal{C}$, without fixing the indexing poset. Specifically, this category, denoted by $Pers(\mathcal{C})$, has for objects pairs $(P, F : P \to \mathcal{C})$ where P is a poset (*i.e.* F is a P-persistent $\mathcal{C}$-object). A morphism from $(P, F : P \to \mathcal{C})$ to $(Q, G : Q \to \mathcal{C})$ is a pair (ϕ, σ) where ϕ is a morphism $P \to Q$ of **Pos** and σ is a natural transformation $F \Rightarrow G \circ \phi$:

$$\begin{tikzcd}
P \arrow[rr, "\phi"] \arrow[dr, "F"'] \arrow[rr, phantom, "\overset{\sigma}{\Nearrow}"', shift right=4] & & P' \arrow[dl, "G"] \\
 & \mathcal{C} &
\end{tikzcd}$$

Composition of morphisms is given by $(\psi, \tau) \circ (\phi, \sigma) = (\psi \circ \phi, \tau_\phi \circ \sigma)$, and the identity on (P, F) is the pair $(1_P, 1_F)$. We denote by $IsoPers(\mathcal{C})$ the subcategory of $Pers(\mathcal{C})$ in which we only take morphisms (ϕ, σ) such that σ is a natural isomorphism. Compare this with the category of natural systems, recalled in Section 11.2.1.

We say that a category $\mathcal{C}$ is *poset complete* (resp. *poset co-complete*) if any functor $\Phi \to \mathcal{C}$, where Φ is a poset, has a limit (resp. colimit) in $\mathcal{C}$. The following result states that when $\mathcal{C}$ is poset (co-)complete, we can reconstitute a diagram on a poset via a colimit on the poset itself.

Let P be a poset and consider an inclusion $G : \Phi \to \mathbf{Pos}_{in}$, where Φ is a category of sub-posets of P and inclusions, such that $\underset{\Phi}{\mathrm{colim}} G = P$. Suppose that we are given a functor $D : P \to \mathcal{C}$, *i.e.* an object of $Pers(\mathcal{C})$. Since the colimit of G is P, for each p in Φ, we have an inclusion $i_p : P_p \to P$. Using these inclusions, we define a functor $F : \Phi \to Pers(\mathcal{C})$, sending each p to the restriction of D to P_p, *i.e.* $F_p := (i_p)^* D$. We denote by $(P_p, \leq_p)$ the domain of F_p. When $p \leq_\Phi p'$, we obtain a morphism $(F^1_{p,p'}, F^2_{p,p'})$ of $Pers(\mathcal{C})$. Note that F is a functor $F : \Phi \to IsoPers(\mathcal{C})$, since $F_{p'}$ is the restriction of D to $P_{p'}$, hence agree with F_p, which is the restriction of D to $P_p \subseteq P_{p'}$.

13.3.5. Proposition. *Let P, G, D and F be as defined in the above paragraph. If $\mathcal{C}$ is poset co-complete, we have*

$$\underset{Pers(\mathcal{C})}{\mathrm{colim}} F = D.$$

Proof. The functor F determines a functor G_F from Φ to $\mathbf{Pos}$:

$$G_F : \Phi \longrightarrow \mathbf{Pos}$$

$$\begin{tikzcd}
p \arrow[r, mapsto] \arrow[d, "\leq_\Phi"'] & P_p \arrow[d, "F^1_{p,p'}"] \\
p' \arrow[r, mapsto] & P_{p'}
\end{tikzcd}$$

By hypothesis, we have $G_F = G$, so the colimit of G_F is equal to P. In this optic, elements of P are equivalence classes $[x]$ of elements $x_p \in P_p$ under the equivalence relation generated by $x_p \sim x_{p'}$ if, and only if, $p \leq_\Phi p'$ and $F^1_{p,p'}(x_p) = x_{p'}$. Notice that if $[x_p] = [x_{p'}]$, there exists a zig-zag of $F^1_{q,q'}$'s between them. Therefore, since the components $F^2_{q,q'}$'s are all natural isomorphisms, we have $F_p(x_p) \simeq F_{p'}(x_{p'})$, hence

$$[x_p] = [x_{p'}] \Rightarrow F_p(x_p) \simeq F_{p'}(x_{p'}) \tag{13.3.1}$$

Furthermore, we know that $[x] \leq_P [y]$ if, and only if, there exists a finite sequence $p = p_0, \ldots, p_n = p'$ and elements v_i, w_i of P_{p_i} such that

$$x_p = v_{p_0} \leq_{p_0} w_{p_0} \simeq v_{p_1} \leq_{p_1} w_{p_1} \simeq \cdots \simeq v_{p_{n-1}} \leq_{p_{n-1}} w_{p_{n-1}} \simeq v_{p_n} \leq_{p_n} w_{p_n} = y_{p'}, \tag{13.3.2}$$

and $[x_p] = [x]$, $[y_{p'}] = [y]$.

We also point out that each equivalence class $[x]$ of P is endowed with an order which is inherited from Φ: $x_p \leq_{[x]} x_{p'}$ if, and only if, $p \leq_\Phi p'$. For each of these equivalence classes, we consider a functor

$$F_{[x]} : [x] \longrightarrow \mathcal{C}$$

$$\begin{tikzcd}
x_p \arrow[r, mapsto] \arrow[d, "\leq_{[x]}"'] & F_p(x_p) \arrow[d, "F^2_{p,p'}(x_p)"] \\
x_{p'} \arrow[r, mapsto] & F_{p'}(x_{p'})
\end{tikzcd}$$

By hypothesis, $\mathcal{C}$ is co-complete, so we obtain the colimit $c_{[x]}$ in $\mathcal{C}$ of the diagram given by $F_{[x]}$. For every $x_p, x_{p'}$, $F_p(x_p) \cong F_{p'}(x_{p'})$ since the natural transformations induced by F are isomorphisms. Thus

$$F_p(x_p) \cong c_{[x]} \text{ for any } x_p \in [x] \tag{13.3.3}$$

Now suppose that $[x] \leq [y]$. We want to prove that $c_{[y]}$ is a co-cone for $F_{[x]}$. We reason by induction on the length n of the sequence given in Equation (13.3.2). If $n = 0$, Equation (13.3.2) amounts to the existence of $p_0 \in \Phi$ and representatives $v_{p_0}, w_{p_0} \in P_{p_0}$ with $v_{p_0} \leq_{p_0} w_{p_0}$, $p = p_0 = p'$, $x_p = v_{p_0}$, $w_{p_n} = y_{p'}$ and $[x_p] = [x]$, $[y_{p'}] = [y]$. We must show that for all $x_p \leq_{[x]} x_{p'}$, we have a commutative diagram of the following shape:

$$\begin{tikzcd}
F_p(x_p) \arrow[rr, "F^2_{p,p'}(x_p)"] \arrow[dr] & & F_{p'}(x_{p'}) \arrow[dl] \\
& c_{[y]} &
\end{tikzcd} \tag{13.3.4}$$

Since $v_{p_0} \leq_{p_0} w_{p_0}$, functoriality of F_{p_0} gives an arrow

$$F_{p_0}(v_{p_0} \leq_{p_0} w_{p_0}) : F_{p_0}(v_{p_0}) \to F_{p_0}(w_{p_0}).$$

This means that we have the following diagram, where all of the morphisms in the top triangle are isomorphisms induced by the natural transformations F^2 (by Equation (13.3.1) in particular):

$$\begin{tikzcd}
F_p(x_p) \arrow[r, dash, "\simeq"] \arrow[rr, bend left, "\simeq"] & F_{p_0}(v_{p_0}) \arrow[r, dash, "\simeq"] \arrow[d] & F_{p'}(x_{p'}) \\
& F_{p_0}(w_{p_0}) \arrow[d] & \\
& c_{[y]} &
\end{tikzcd}$$

Therefore $c_{[y]}$ is a co-cone for $F_{[x]}$ and we obtain a unique arrow $\Gamma_{[x],[y]} : c_{[x]} \to c_{[y]}$ in $\mathcal{C}$.

Suppose now that we have the property that $c_{[y]}$ is a co-cone for $F_{[x]}$ when the length of the sequence in Equation (13.3.2) is strictly below n. We want to prove the property still holds when the sequence has length n. In the latter case, we have a finite sequence $p = p_0, \ldots, p_n = p'$ and elements v_i, w_i of P_{p_i} such that

$$x_p = v_{p_0} \leq_{p_0} w_{p_0} \simeq v_{p_1} \leq_{p_1} w_{p_1} \simeq \cdots \simeq v_{p_{n-1}} \leq_{p_{n-1}} w_{p_{n-1}} \simeq v_{p_n} \leq_{p_n} w_{p_n} = y_{p'},$$

and $[x_p] = [x]$, $[y_{p'}] = [y]$, and we suppose that $c_{[w_{p_{n-1}}]}$ is a co-cone for $F_{[x]}$.

By Equation (13.3.3), we have $c_{[w_{p_{n-1}}]} \cong F_{p_{n-1}}(w_{p_{n-1}})$, and since $w_{p_{n-1}} \simeq v_{p_n}$, we also have $F_{p_{n-1}}(w_{p_{n-1}}) \cong F_{p_n}(v_{p_n})$ by Equation (13.3.1). Finally, since $v_{p_n} \leq_{p_n} w_{p_n} = y_{p'}$, we have a morphism $F_{p_n}(v_{p_n}) \to F_{p'}(y_{p'})$ in $\mathcal{C}$ induced by $F_{p'}$. So we have the following:

$$\begin{tikzcd}
F_p(x_p) \arrow[r, dash, "\simeq"] \arrow[rr, bend left, "\simeq"] & F_{p_0}(x_{p_0}) \arrow[r, dash, "\simeq"] \arrow[d] & F_{p'}(x_{p'}) \\
& F_{p_n}(v_{p_n}) \arrow[d] & \\
& F_{p'}(y_{p'}) \arrow[d] & \\
& c_{[y]} &
\end{tikzcd}$$

Therefore $c_{[y]}$ is again a co-cone for $F_{[x]}$, giving the unique arrow $\Gamma_{[x],[y]} : c_{[x]} \to c_{[y]}$ in $\mathcal{C}$.

We define a functor $\Gamma : P \to \mathcal{C}$ which sends $[x]$ to $c_{[x]}$ and $[x] \leq [y]$ to the arrow $\Gamma_{[x],[y]}$ given by the argument above. It is clear that $\Gamma = D$; it remains to show that Γ is indeed the colimit of F.

Let $D' : p' \to \mathcal{C}$ another co-cone for F. For every $p \in \Phi$, we have morphisms (j^1_p, j^2_p) such that the following diagram commutes in $Pers(\mathcal{C})$ for all $p \leq p'$:

$$\begin{tikzcd}
F_p \arrow[rr, "{(F^1_{p,p'}, F^2_{p,p'})}"] \arrow[dr, "{(j^1_p, j^2_p)}"'] & & F_{p'} \arrow[dl, "{(j^1_{p'}, j^2_{p'})}"] \\
& D &
\end{tikzcd}$$

In particular, this means we have the following commutative diagram in **Pos**:

$$\begin{tikzcd}
P_p \arrow[rr, "F^1_{p,p'}"] \arrow[dr, "j^1_p"'] & & P_{p'} \arrow[dl, "j^1_{p'}"] \\
& P' &
\end{tikzcd}$$

This in turn means that P' is a co-cone over G, so we obtain a unique morphism $\phi : P \to P'$. Furthermore, D' being a co-cone over F implies that for all $x_p, x_{p'} \in [x]$, we have the following commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd}
F_p(x_p) \arrow[rr, "F^2_{p,p'}(x_p)"] \arrow[dr, "j^2_p(x_p)"'] & & F_{p'}(x_{p'}) \arrow[dl, "j^2_{p'}(x_{p'})"] \\
& D'(j^1_p(x_p)) = D'(j^1_{p'} \circ F^1_{(p,p')(x_p)}) &
\end{tikzcd}$$

Since $c_{[x]}$ is the colimit of $F_{[x]}$, we therefore obtain a unique arrow $\sigma_{[x]} : c_{[x]} \to D'(j^1_p(x_p))$ for any representative $x_p \in [x]$. Denoting by $i_p : P_p \to P$ the morphisms in **Pos** induced by the colimit of G, we have that $\phi([x]) = \phi(i_p(x_p)) = j^1_p(x_p)$ for any $p \in \Phi$ such that P_p contains a representative of $[x]$. With some diagram chasing, we find that (ϕ, σ) is a morphism of $Pers(\mathcal{C})$ from Γ to D', and by construction is the unique map making the colimit triangles commute. Thus D is the colimit of F in $Pers(\mathcal{C})$. ☐

13.3.6. Natural homology as a colimit. Using the above proposition, we obtain the following theorems:

13.3.7. Theorem. *Let $\mathcal{X} = (X, dX)$ be a pospace and α a point in X.*

- **i)** *The natural homology of $\mathcal{X}$ is the colimit in $Pers(\mathbf{Vect}_{\mathbb{K}})$ of the persistent homology along each of its traces.*
- **ii)** *The natural homology of $\mathcal{X}$ is the colimit in $Pers(\mathbf{Vect}_{\mathbb{K}})$ of the persistent homology its of its maximal traces, seen as chains in $\mathbb{P}(\mathcal{X})$, completed with pullbacks (resp. quasi-pullbacks).*
- **iii)** *The natural homology of the up-set of α, seen as a constant trace, in $\mathbb{P}(\mathcal{X})$ is the colimit in $Pers(\mathbf{Vect}_{\mathbb{K}})$ of the persistent homologies of the traces passing through α or of the maximal chains passing through α completed with pullbacks or quasi-pulbacks.*

Proof. These are direct consequences of Propositions 13.3.5 and 13.3.2. ☐

This method is applicable to any colimit diagram in **Pos** given by inclusions into the maximal chains. Indeed, we have the persistent homology of the maximal chains and we can induce maps on any subposets thereof by restriction, thereby inducing maps in $Pers(\mathbf{Vect}_{\mathbb{K}})$ constituting the corresponding cone in the functor category.

CHAPTER 14.
CONCLUSION AND PERSPECTIVES

Here we summarise the work and presenting perspectives on future work. Shortly put, the main contributions of this thesis project are the following:

- **i)** The introduction of an algebraic setting for coherence proofs by rewriting, which lends itself towards formalisation [17].
- **ii)** The provision of an algebraic coherence theorem for abstract rewriting systems [16], which inspired a coherence theorem for abstract rewriting systems in any dimensions.
- **iii)** A contribution to a project of formalisation of generalised higher categories and their power-set liftings [14, 18].
- **iv)** A refinement of directed algebraico-topological invariants, notably in terms of time-reversal and relative homotopy theory [15].
- **v)** The establishment of a concrete link between natural homology and persistence theory.

In **Part I**, first steps towards a formalisation of higher coherence theorems were made. First, coherence in abstract rewriting, even in higher dimensions, was shown to be essentially two-dimensional via the notion of underlying rewriting polygraph. By extending Kleene algebra to higher structures encoding the paving mechanisms offered by higher dimensional rewriting theory, we provide an algebraic setting for coherence proofs. We formulate and prove a coherence theorem for (higher) abstract rewriting systems, in which diagrammatic reasoning is replaced by a series of deductions in a simple algebraic signature, and check this formalisation using power-set models of free higher categories. Finally, we push this correspondence to its limits, showing a Jónsson-Tarski style duality for generalised categories, *i.e.* catoids, and higher Kleene algebras with extra structure, *i.e.* higher quantales.

The first results presented in **Part II** concern the problem of time-symmetry for natural homology and homotopy on the one hand, and the theory of relative natural homotopy on the other. The first point was accomplished by equipping these directed invariants with a natural notion of composition, which in turn allows their interpretation as certain categories. Opposition in these categories is then shown to be the invariant associated to the time-reversal of the considered space. For the second point, we use notions of exactness in non-homological categories to show that the long exact sequence given by classical relative homotopy extends to natural homotopy in a way which interacts well with the notion of fibration. This refinement of natural homotopy and homology allow a finer analysis of directed spaces.

Finally, **Part II** presents results linking natural homology to the domain of persistence. This provides a first step towards tractable algebraic invariants for directed spaces. The notion of evolution of (classical) homological type, present in persistence by design and in directed topology by structure, provided a first foothold in comparing these approaches. This was strengthened by the observation that the domain of the natural homotopy functor is in fact a poset when considering partially ordered spaces. Furthermore, the main structural component of directed spaces, directed paths, were shown to generate uni-dimensional persistence modules, which may then be amalgamated to recover natural homology. This not only provides a method for calculating (subdiagrams of) natural homology, but also provides insight into its structure.

Perspectives and future work

Just as in the case of the results obtained over the course of this thesis project, the avenues of future work focus on algebraico-topological aspects of higher directed structures and the formalisation thereof. These are described in more detail below, but are summed up as follows:

i) A continuation of the formalisation of higher rewriting systems in the proof assistant Isabelle.

ii) Extending the algebraic formulation of coherence to capture polygraphic resolution, leading to a formal approach to the calculation of cofibrant replacements by rewriting.

iii) Describing string rewriting and other paradigms of algebraic rewriting via Kleene algebraic or quantalic structures.

iv) Defining a higher homotopical invariant for directed spaces.

v) Introducing a notion of ω-directed space.

vi) Exploring the consequences the strong connection between persistence theory and directed homology.

Formalisation of coherence. As higher categorical structures become more complex, and in particular dependant on complex coherence conditions, the development of formal tools for checking these conditions becomes primordial not only in exploring the mathematical properties of such structures, but also in their practical use. The combinatorics involved in such higher settings become unwieldy in human hands due to the complexity of weak higher structures - a look at the coherence conditions for bi-categories, for example, demonstrates this readily. The use of interactive theorem provers translates such routine, but humanly untractable, (diagrammatic) verifications into first order logic, thereby allowing machines to generate counter-examples or verify certain properties. The introduction of higher algebraic structures such as higher Kleene algebras and higher quantales provides an algebraic setting which lends itself to this type of formalisation. Furthermore, the introduction of algebraic structures with total multi-operations generalising higher categories not only facilitates their formalisation, but has also offered insight into their structure, for example in the weakness of homomorphism laws.

In short, this algebraic treatment of higher directed structures provides a first step toward the development of formal tools, while encoding them in theorem provers like Isabelle has pushed us to precisely formulate their structure and eliminate axiomatic redundancies. Continuing this line of research, next natural steps include the following:

Isabelle theories. Firstly, the Isabelle repository, to which the author has contributed, will be completed to include results related to coherence found in this thesis. While the Isabelle theories for higher Kleene algebras, higher quantales and higher catoids have been established, the coherence theorems will be encoded therein, including the consistency check via the Jónsson-Tarski correspondence theorem. In short, Theorems 9.3.2, 9.2.9, and 9.4.3, as well as necessary structural lemmas, will be formalised.

Resolutions and cofibrance. The natural next step is to capture the iterated coherence procedure described in Section 3.2.3. Indeed, while the first step of this procedure is assured by Theorem 8.5.2, but we have yet to obtain the necessary result for propagation to higher dimensions. In more detail, consider a higher Kleene algebra K and a local confluence filler A for an element ϕ. Given a normalisation strategy σ associated to ϕ, it must be shown that the element $\sigma\overline{\sigma}$ is a section of normal forms associated to the completion of A, seen as a (higher) convergent rewriting system.

Such a result would provide a means of describing the notion of *polygraphic resolution*, as described in [91], and the construction of such a resolution by convergent rewriting techniques, see [64]. This can in turn be used to construct cofibrant replacements of (higher) algebraic structures.

Formalisation of algebraic rewriting. In this work, we have algebraically captured coherence mechanisms for abstract rewriting systems, but have yet to push this analysis to the domain of string rewriting systems. As described in Section 5.3, these are (two-dimensional) rewriting systems which take the algebraic operations of the presented structure into account.

Even consistency results for SRS have not been captured in any other setting than that of polygraphic rewriting, see [68]. This is essentially due to the notion of *contexts*, of which whiskers (see Section 5.3.7) are a special case, which allow the definition of critical branchings. One reason we began considering quantalic structures such as those presented in Chapter 10 was in order to use the continuity of multiplication to obtain a residuation operation, described in the case of relation algebras in [30], allowing the "cutting away" of contexts. Describing critical branchings in higher quantales would lead to an algebraic formulation of the critical branching lemma [68], see Theorem 5.3.16, and such a formulation of critical coherence, see Theorem 7.2.1. This reduces the consistency or coherence checks drastically.

Directed algebraic topology. Algebraic topology, having originated in the undirected paradigm of classical topology, has birthed category theory, which is naturally equipped with (higher) directed cells. Open problems in higher category theory, for example the Simpson conjecture, seem to be strongly related to the homotopy theory of directed spaces in the form of, for example, the directed homotopy hypothesis. Such considerations equally apply to (directed) homotopy type theory. Without a topological notion of higher direction, developing a sound homotopy theory for (higher) directed structures remains nebulous[1]. Indeed, the homotopical invariant studied in this thesis, natural homotopy, is in some sense a topological approach to studying the homotopy of (weak) $(\infty, 1)$-categories. This is due to the fact that directed spaces are in essence encoded by the one-dimensional information given by paths. Without a higher notion of directed space, there is no topological guide to understand what appropriate directed homotopical invariants should be.

To summarise, it is my belief that a fundamental problem in modern mathematics stems from directed structures having their basis in an undirected notion, namely homotopy for classical spaces. Constructing higher categorical structures encoding the homotopy of directed spaces and developing a notion of higher directed space present future avenues of work which on the one hand extend this thesis and on the other hand confront the clash between directedness and classical topology. We sum this up in the points below:

Higher homotopical invariants. In Section 12.1.2 of this document, it was shown that natural homotopy and homology functors are equipped with composition pairings, resulting in Theorem 12.1.5, which states that there is a functorial assignment of a 1-category $\mathcal{C}^n_{\mathcal{X}}$ to the n^{th} natural homotopy functor $\overrightarrow{P}_n(\mathcal{X})$. Using a similar construction, we hope to build a (weak) $(\infty, 1)$-category capturing all of this information. Indeed, since each of the categories $\mathcal{C}^n_{\mathcal{X}}$ is above $\overrightarrow{\mathbf{P}}(\mathcal{X})$, the trace category, and since the 0-cells, *i.e.* points of X, are conserved, the globularity of higher categories should provide a method by which to "stack up" the information in each natural homotopy category $\mathcal{C}^n_{\mathcal{X}}$ into a single higher structure. Indeed, by construction, the considered homotopies are equally globular.

A notion of higher directed space. The above, however, does not address the notion of higher directed space. In recent conversations between F. Paugam and the author, a notion of higher directed space has been discussed, based on *complicial sets*. These were first introduced by Roberts [100], made more precise via the Street nerve [107], and finally developed by Verity, see for example [115]. These are more general versions of *weak Kan complexes*, combinatorial models of weak $(\infty, 1)$-categories. Roughly speaking, complicial sets consist of a simplicial set along with a subcomplex of "marked" simplices. These marked simplices differentiate higher cells which encode structure on lower dimensions from directed higher cells.

Given a directed space $\mathcal{X} = (X, dX)$, we can construct the singular simplicial complex $\Pi_\infty(X)$ associated to its underlying space. The idea is to recover the directed space by restricting $\Pi_1(X)$ to the paths found in dX, marking some of them as equivalences, and then mark all higher cells, thereby obtaining a complicial complex associated to

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[1] see John Baez's tweet on the subject.

$\mathcal{X}$. Pushing this further, a notion of ω-directed space on a topological space X could be obtained by considering a complicial complex which is a subcomplex of $\Pi_\infty(X)$. In other words, we would obtain a directed space in which homotopies are themselves also directed. This would provide a higher directed topological setting and provide insight into what homotopical invariants should be for such structures.

Homology theories. In this thesis, we focussed on the computation of natural homology via tractable methods offered by persistence theory. While this algorithmic computational aspect will continue to develop, the link that was established between these two theories leads to several natural questions. Firstly, using the tractable algorithmic approach offered by uni-dimensional persistence could lead to tractable invariants for directed spaces. Secondly, understanding how to interpret barcodes in the case of natural homology could be approached via the colimit construction on chains; this is linked to open questions in multi-dimensional persistence theory. Next, the link between interleaving distance, a notion from persistence theory, and bisimulation of natural systems will be explored. Finally, I believe that the notion of composition pairing, or some such operation, could be interpreted in persistence theory and have links to hybrid and dynamical systems.

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